

IDEMPOTENT SPLITTINGS, COLIMIT COMPLETION, AND WEAK ASPECTS OF THE THEORY OF MONADS

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ABSTRACT. We show that some recent constructions in the literature, named ‘weak’ generalizations, can be systematically treated by passing from 2-categories to categories enriched in the Cartesian monoidal category of Cauchy complete categories.

The word “weak” has been used in category theory with various meanings. An early example was the weakening of universal properties to ask only for the existence of a map, not the existence and uniqueness; this gives, for example the notion of a *weak limit* of a diagram: a cone through which every other cone factorizes.

Often, but not always, such weak limits arise in situations which are either explicitly or implicitly homotopical, and even though one might not have uniqueness of factorizations, all such factorizations might be homotopic: here we have in mind examples like the homotopy category (of spaces) and the category of projective modules over a ring R . Then again, the notion of homotopy might arise because one is working not just in a category but in a 2-category (or bicategory). This case led to a second use of the word, coming from the theory of higher categories: one speaks of weak categories, or weak functors, or weak limits, to mean notions where all equations are expected to hold only “up to higher homotopy”. For instance one has the theory of *weak n -categories*.

The weakness considered in this paper is somewhat different, and originated in the theory of Hopf algebras. In the 1990s, questions arising in various areas of mathematics and even physics suggested the need for a generalization of Hopf algebras. In subfactor theory, for example, the description of reducible inclusions required a new symmetry structure. Meanwhile in topology, invariants of 3-manifolds were constructed that could not be derived from Hopf algebras. In some low dimensional quantum field theories, non-integral valued quantum dimensions occurred, implying that the internal symmetry could not be described by a Hopf algebra. These phenomena led to the introduction of *weak Hopf algebras* [7] which successfully dealt with all these questions. Apart from Hopf algebras themselves, examples of weak Hopf algebras include Hayashi’s face algebras [10], Yamanouchi’s (finite dimensional) generalized Kac algebras [23], Ocneanu’s paragroups [17], and in particular finite groupoid algebras and their linear duals.

Large parts of the theory of Hopf algebras have now been generalized to the weak context — we could point, for instance, to the study of Galois extensions by weak Hopf algebras in [9, 8, 1], involving a non-commutative version of principal bundles of structure *groupoids* (as opposed to the structure groups of the non-weak theory), and to the study of weak Hopf algebras in braided monoidal categories [2, 18].

A weak Hopf algebra H (over a commutative ring k) involves, like a Hopf algebra, a k -module equipped with a k -algebra structure and a k -coalgebra structure, subject to compatibility conditions. The difference is in the compatibility conditions: although in a weak Hopf algebra the comultiplication $H \rightarrow H \otimes H$ will still preserve the multiplication, the condition that it preserve the unit is replaced by a weaker one; similarly the multiplication $H \otimes H \rightarrow H$ will still preserve the comultiplication but need not preserve the counit.

The axioms of a weak Hopf algebra ensure that its category of representations is monoidal. However, the tensor product of representations is not their k -module tensor product (as it is in the case of a Hopf algebra) but a certain k -module retract of it, obtained by splitting a certain idempotent.

In the last two paragraphs, we have already seen two key aspects of weak Hopf algebras: the weakening of unit conditions, and the splitting of idempotents. In fact these two are very tightly related, especially if we consider identities in categories rather than units in algebras/monoids. Let A and B be categories,

and define a *semifunctor* from A to B to be a morphism of the underlying directed graphs which preserves composition, but which is not required to preserve identities. Then semifunctors from A to B are in natural bijection with functors from A to the category QB obtained from B by freely splitting the idempotents. We recall the construction and properties of QB in Section 1.1 below. The process of freely splitting idempotents in a category is often called *Cauchy completion*, because it is the case $\mathcal{V} = \mathbf{Set}$ of a general construction on enriched categories which in the case where \mathcal{V} is Lawvere’s category $[0, \infty]$ gives the Cauchy completion of a (generalized) metric space [15].

We then go on to develop a whole “weak world”, parallel to the classical world. At the risk of oversimplifying somewhat, we could summarize the approach by saying that any classical notion implemented in a 2-category \mathcal{K} should be applied not to \mathcal{K} itself; rather one should first take the 2-category $Q_*\mathcal{K}$, obtained from \mathcal{K} by taking the Cauchy completion of the hom categories, and then apply the notion there. For example, as in Section 2.1 below, we can regard a (strict) monoidal category B as a one object 2-category. Performing local Cauchy completion, we obtain a monoidal structure on the Cauchy completion QB of B , and one can now consider monoids not in B but rather in QB . This will be our notion of “weak monoid”.

In fact because of the variety of meanings of the epithet weak, we have decided not to use it as our general naming device; instead we use the prefix “demi-”, so in the case of the previous paragraph, we define a *demonoid* in a monoidal category B to be a monoid in QB .

We gradually work through various other structures, weakening them as we go. This includes monads and their algebras in Section 2, and limits in Section 4. The most important instance of a limit for our purposes is that of an Eilenberg-Moore object in Section 4.2. Our ultimate goal in Section 6 is to develop a weak version of the formal theory of monads [19, 14], building on the start made in [4]. In particular, we see that for a 2-category \mathcal{K} in which idempotent 2-cells split, the 2-category $\mathbf{EM}^w(\mathcal{K})$ of [4] is the free completion of \mathcal{K} , as a 2-category in which idempotent 2-cells split, under (bicategorical) Eilenberg-Moore objects: see Corollary 5.3.

The classical formal theory of monads has applications in Hopf algebra theory. Recall that a bialgebra (over a field) can be regarded as an opmonoidal monad in the monoidal category of vector spaces (considered as a single object bicategory). Because of this, the Eilenberg-Moore category of algebras (i.e. the category of modules over the bialgebra) is monoidal with the monoidal structure lifted from the category of vector spaces (so that the forgetful functor is strict monoidal). Moreover, any monad in the Eilenberg-Moore category (i.e. module algebra over the bialgebra) induces a wreath in the category of vector spaces. The corresponding wreath product is called the ‘smash product algebra’.

In a similar way, our weak version occurs in constructions related to weak Hopf algebras. Weak bialgebras (again, over a field) are ‘weak bimonads’ in the monoidal category of vector spaces. Weak bimonads were studied in [5]. They are monads equipped with the additional structure which ensures that the Eilenberg-Moore category is monoidal such that the forgetful functor possesses a so-called separable Frobenius monoidal structure in the sense of [22]. In this case the monoidal structure of the Eilenberg-Moore category is weakly lifted from the category of vector spaces in the sense discussed in the current paper. Moreover, any monad in the Eilenberg-Moore category (i.e. any module algebra over a weak bialgebra) induces a weak wreath in the category of vector spaces; the corresponding weak wreath product in the sense of this paper is the weak smash product algebra.

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1. LOCAL CAUCHY COMPLETION

In later sections of this paper we will perform some constructions in Cauchy completions of categories, i.e. categories obtained by freely splitting idempotents in some category. We start with collecting some results about Cauchy completions and by illustrating the relevance of this process for our subject.

1.1. The Cauchy completion functor $\mathbf{cat} \rightarrow \mathbf{cat}$. Write \mathbf{cat} for the category of categories and functors; later we shall want to consider this also as a 2-category \mathbf{Cat} . Define a semicategory to be a directed graph with an associative composition (no identities assumed), and a semifunctor as the obvious notion of homomorphism of semicategories. Thus a category is precisely a semicategory with identities, and a functor is precisely an identity-preserving semifunctor between categories. Write \mathbf{scat} for the category of semicategories and semifunctors.

The evident forgetful functor $U : \mathbf{cat} \rightarrow \mathbf{scat}$ of course has a left adjoint $F : \mathbf{scat} \rightarrow \mathbf{cat}$ which freely adjoins identities to a semicategory. But it also has a right adjoint $R : \mathbf{scat} \rightarrow \mathbf{cat}$ which picks out all “potential identities” in the form of idempotents. Explicitly, for a semicategory S , the objects of RS are the idempotents $\sigma : s \rightarrow s$ in S , and a morphism $(s, \sigma) \rightarrow (t, \tau)$ in RS is a morphism $\varphi : s \rightarrow t$ in S with $\tau\varphi = \varphi = \varphi\sigma$. The identity on (s, σ) is just σ .

The adjunction $U \dashv R$ induces a monad on \mathbf{cat} . We write $Q = RU$ for the endofunctor, and $q : 1 \rightarrow Q$ for the unit. For any category B , the component $q : B \rightarrow QB$ of the unit exhibits QB as the Cauchy completion of B : the category obtained by freely splitting the idempotents of B . The functor $Q = RU$ is right adjoint to FU and so preserves all limits in \mathbf{cat} .

Thus for categories A and B , a semifunctor from A to B is the same as a functor from A to QB . That is, the Kleisli category of the monad Q on \mathbf{cat} can be regarded as the category of generalized functors which are no longer compatible with identity morphisms. More generally, we shall see that many weak notions can be obtained by first applying Q , then considering the usual notion.

1.2. The Cauchy completion 2-functor $\mathbf{Cat} \rightarrow \mathbf{Cat}$. Of course there is also a 2-category \mathbf{Cat} of categories, functors, and natural transformations, and Q extends to a 2-monad on \mathbf{Cat} . As a 2-functor, Q is not a right adjoint, and does not preserve 2-categorical limits in general, although it does preserve products and equalizers, and so all conical limits. Of particular importance will be the fact that Q preserves finite products.

Example 1.1. Let $\mathbf{2}$ be the category consisting of two objects and a single non-identity arrow between them. There is a 2-categorical limit A^2 called the **2-power** of A , defined by the universal property

$$\mathbf{Cat}(X, A^2) \cong \mathbf{Cat}(X, A)^2$$

where the right hand side is just the category of arrows in $\mathbf{Cat}(X, A)$. Explicitly, A^2 is just the category of functors from $\mathbf{2}$ to A . The 2-functor Q does not preserve the power A^2 strictly: the canonical comparison functor $Q(A^2) \rightarrow Q(A)^2$ is not invertible, although it is a surjective equivalence.

Indeed, an object of $Q(A^2)$ is an arrow $\alpha : a_1 \rightarrow a_2$ in A , along with idempotents $\alpha_1 : a_1 \rightarrow a_1$ and $\alpha_2 : a_2 \rightarrow a_2$ satisfying $\alpha_2\alpha = \alpha\alpha_1$. An arrow in $Q(A^2)$ from $(\alpha, \alpha_1, \alpha_2)$ to $(\beta, \beta_1, \beta_2)$ consists of arrows $\varphi_i : a_i \rightarrow b_i$ for $i \in \{1, 2\}$, satisfying $\beta\varphi_1 = \varphi_2\alpha$ and $\beta_i\varphi_i = \varphi_i = \varphi_i\alpha_i$ for $i \in \{1, 2\}$.

An object of $Q(A)^2$ consists of idempotents $\alpha_i : a_i \rightarrow a_i$ for $i \in \{1, 2\}$, and a morphism $\alpha' : a_1 \rightarrow a_2$ satisfying $\alpha_2\alpha' = \alpha' = \alpha'\alpha_1$; note the extra condition compared to $Q(A^2)$. A morphism in $Q(A)^2$ from $(\alpha', \alpha_1, \alpha_2)$ to $(\beta', \beta_1, \beta_2)$ consists of morphisms $\varphi_i : a_i \rightarrow b_i$ for $i \in \{1, 2\}$ satisfying $\beta'\varphi_1 = \varphi_2\alpha'$ and $\beta_i\varphi_i = \varphi_i = \varphi_i\alpha_i$.

The comparison functor $K : Q(A^2) \rightarrow Q(A)^2$ is the image of the identity functor under the composite map

$$\mathbf{Cat}(A^2, A^2) \xrightarrow{\cong} \mathbf{Cat}(A^2, A)^2 \xrightarrow{Q} \mathbf{Cat}(Q(A^2), Q(A))^2 \xrightarrow{\cong} \mathbf{Cat}(Q(A^2), Q(A)^2).$$

Its effect is not perhaps what one might expect: it sends an object $(\alpha, \alpha_1, \alpha_2)$ of $Q(A^2)$ to $(\alpha\alpha_1, \alpha_1, \alpha_2)$, while it sends a morphism (φ_1, φ_2) to (φ_1, φ_2) . (In brief, set $\alpha' = \alpha\alpha_1$.) Clearly K is faithful; to see that it is full we must check that for a morphism (φ_1, φ_2) in $Q(A)^2$ we have $\beta\varphi_1 = \varphi_2\alpha$, but $\beta\varphi_1 = \beta\beta_1\varphi_1 = \beta'\varphi_1 = \varphi_2\alpha' = \varphi_2\alpha\alpha_1 = \varphi_2\alpha_2\alpha = \varphi_2\alpha$ as required. Thus K is fully faithful; it is also clearly surjective on objects, and so an equivalence of categories, but is not injective on objects. (For instance, if $\alpha_1 : a_1 \rightarrow a_1$ is any non-identity idempotent, let $a_2 = a_1$ and $\alpha_2 = \alpha_1$, and then $\alpha = \alpha_1$ and $\alpha = 1$ give two different objects of $Q(A^2)$ which get sent by K to the same object of $Q(A)^2$.)

Example 1.2. Similarly, for any category C , there is a 2-categorical limit A^C called the C -power of A , defined by $\mathbf{Cat}(X, A^C) \cong \mathbf{Cat}(X, A)^C$. In general, Q does not preserve such powers, even up to equivalence.

This can be seen as follows. The functor $q : A \rightarrow QA$ induces a fully faithful map $\mathbf{Cat}(C, A) \rightarrow \mathbf{Cat}(C, QA)$, and $\mathbf{Cat}(C, QA)$ is Cauchy complete since QA is; thus there is an induced fully faithful inclusion $Q(\mathbf{Cat}(C, A)) \rightarrow \mathbf{Cat}(C, QA)$, and this is the canonical comparison map $Q(A^C) \rightarrow (QA)^C$. It is an equivalence if and only if every $f : C \rightarrow QA$ is a retract of some functor of the form $qg : C \rightarrow QA$, where $g : C \rightarrow A$.

Let A be the free-living idempotent, consisting of a single object $*$ and a single non-identity arrow e satisfying $e^2 = e$, and let $C = QA$: this has objects 1 and e . We shall show that the identity functor $1 : QA \rightarrow QA$ is not a retract of a functor of the form qg , where $g : QA \rightarrow A$. To give a functor $QA \rightarrow A$ is equivalently to give a split idempotent in A . But the only idempotent which splits in A is the identity. Thus g would have to be the map constant at the unique object $*$ of A . Now any retract of qg would have to be defined using retracts of the object 1 of QA , but it has no non-trivial retracts, and so qg has no non-trivial retracts. In particular the identity functor is not a retract.

1.3. The local Cauchy completion 2-functor $2\text{-Cat} \rightarrow 2\text{-Cat}$. Since $Q : \mathbf{Cat} \rightarrow \mathbf{Cat}$ preserves finite products, it induces a 2-functor $Q_* : 2\text{-Cat} \rightarrow 2\text{-Cat}$ sending a 2-category \mathcal{K} to the 2-category $Q_*\mathcal{K}$ with the same objects, obtained by applying Q to each hom-category.

2. MONADS

Monads (A, t) in a 2-category \mathcal{K} are the same as monoids t in the strict monoidal category $\mathcal{K}(A, A)$. A monad in the local Cauchy completion $Q_*\mathcal{K}$ is thus a monoid in $Q_*\mathcal{K}(A, A) = Q(\mathcal{K}(A, A))$. Let us call this a *weak monad* or *demimonad* in \mathcal{K} . These were considered in [4] using the explicit description of Proposition 2.2 below; and also in [24] where the name ‘ η -symmetric regular quasi-monad’ is used.

The 2-natural transformation $q : 1 \rightarrow Q$ induces a 2-natural transformation $q_* : 1 \rightarrow Q_*$, whose component at a 2-category \mathcal{K} is the inclusion 2-functor $q_* : \mathcal{K} \rightarrow Q_*\mathcal{K}$. This sends monads to monads, and so shows how we can regard ordinary monads in \mathcal{K} as demimonads.

2.1. Monoids. Let (B, \otimes, i) be a monoidal category. Since the 2-functor $Q : \mathbf{Cat} \rightarrow \mathbf{Cat}$ preserves finite products, it sends monoidal objects to monoidal objects, and so we obtain a monoidal category (QB, \otimes', qi) , which we usually call $Q(B, \otimes, i)$ or just QB . Explicitly, the tensor product \otimes' on QB is given on objects by $(b, \rho) \otimes' (b', \rho') = (b \otimes b', \rho \otimes \rho')$.

Definition 2.1. A *weak monoid* or *demimonoid* in (B, \otimes, i) is a monoid in $Q(B, \otimes, i)$.

We now make this more explicit as follows.

Proposition 2.2. A demimonoid in B on an object b consists of the following structure:

- (i) an associative multiplication $\mu : b \otimes b \rightarrow b$;
- (ii) a map $\eta : i \rightarrow b$ for which
 - the composite

$$i \cong i \otimes i \xrightarrow{\eta \otimes \eta} b \otimes b \xrightarrow{\mu} b$$

is just η ;

- the composites

$$b \cong b \otimes i \xrightarrow{b \otimes \eta} b \otimes b \xrightarrow{\mu} b \quad \text{and} \quad b \cong i \otimes b \xrightarrow{\eta \otimes b} b \otimes b \xrightarrow{\mu} b$$

are equal (let us call them μ_1);

- the above map $\mu_1 : b \rightarrow b$ satisfies $\mu_1 \mu = \mu$.

PROOF: Given structure as in the proposition, first note that the composite $\mu_1\mu_1$ is given by

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & \mu_1 & & & & \\
 & \nearrow & & \searrow & & & \\
 b & \xrightarrow{\cong} & b \otimes i & \xrightarrow{b \otimes \eta} & b \otimes b & \xrightarrow{\mu} & b \\
 & \searrow & & \nearrow & \searrow & \nearrow & \\
 & & \mu_1 & & \mu & & \\
 & & & & & & b \\
 & & & & & & \downarrow \mu_1 \\
 & & & & & & b
 \end{array}
 \end{array}$$

and so μ_1 is idempotent. Thus (b, μ_1) is an object of QB . The first condition in (ii) says that $\mu_1\eta = \eta$, which in turn says that η is a map in QB from qi to (b, μ_1) . Next we show that μ is a map in QB from $(b, \mu_1) \otimes (b, \mu_1)$ to (b, μ_1) . One half of this is the last requirement in (ii), the other half says that $\mu(\mu_1 \otimes \mu_1) = \mu$; or, equivalently $\mu(\mu_1 \otimes b) = \mu = \mu(b \otimes \mu_1)$. To see the first of these, note that the diagram

$$\begin{array}{ccccccc}
 b \otimes b & \xrightarrow{\cong} & i \otimes b \otimes b & \xrightarrow{\eta \otimes b \otimes b} & b \otimes b \otimes b & \xrightarrow{\mu \otimes b} & b \otimes b \\
 \mu \downarrow & & \downarrow i \otimes \mu & & \downarrow b \otimes \mu & & \downarrow \mu \\
 b & \xrightarrow{\cong} & i \otimes b & \xrightarrow{\eta \otimes b} & b \otimes b & \xrightarrow{\mu} & b
 \end{array}$$

commutes, and that the bottom path is $\mu_1\mu$ which is indeed μ ; the second equation is proved similarly.

Thus we have maps $\mu : (b, \mu_1) \otimes (b, \mu_1) \rightarrow (b, \mu_1)$ and $\eta : qi \rightarrow (b, \mu_1)$ in QB . Associativity is part (i), while the unit laws are the conditions in the second item in (ii), and so these give a monoid in QB .

Conversely, any monoid in QB involves an underlying object (b, μ_1) with μ_1 idempotent, equipped with morphisms $\mu : (b, \mu_1) \otimes (b, \mu_1) \rightarrow (b, \mu_1)$ and $\eta : qi \rightarrow (b, \mu_1)$. Associativity of μ gives (i), the unit laws give the second condition in (ii), the fact that μ is a morphism in QB gives the last condition, and the unit laws and the fact that η is a morphism in QB give the first condition in (ii). \square

Example 2.3. As any adjunction in a 2-category induces a monad, any adjunction in the local Cauchy completion $Q_*\mathcal{K}$ of a 2-category \mathcal{K} induces a demimonad (i.e. a demimonoid in the hom category $\mathcal{K}(A, A)$) which we describe presently.

A 1-cell in $Q_*\mathcal{K}$ is a pair (x, \bar{x}) consisting of a 1-cell x , and an idempotent 2-cell $\bar{x} : x \rightarrow x$, in \mathcal{K} . An adjunction in $Q_*\mathcal{K}$ is given by 1-cells $(x, \bar{x}) : A \rightarrow B$ and $(y, \bar{y}) : B \rightarrow A$ together with 2-cells $\varphi : (x, \bar{x})(y, \bar{y}) \rightarrow (1_B, 1)$ and $\psi : (1_A, 1) \rightarrow (y, \bar{y})(x, \bar{x})$ in $Q_*\mathcal{K}$; obeying the usual triangle identities. Using the triangle conditions

$$\bar{x} = \varphi x . x \psi \quad \text{and} \quad \bar{y} = y \varphi . \psi y,$$

we can express \bar{x} and \bar{y} in terms of φ and ψ . Thus only the normalization conditions $\varphi . \bar{x} y = \varphi = \varphi . x \bar{y}$ and $\bar{y} x . \psi = \psi = y \bar{x} . \psi$ are left. These are equivalent to commutativity of the diagrams

$$\begin{array}{ccc}
 xy & \xrightarrow{x\psi y} & xyxy \\
 & \searrow \varphi & \downarrow \varphi\varphi \\
 & & 1_B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 1_A & \xrightarrow{\psi\psi} & yxyx \\
 & \searrow \psi & \downarrow y\varphi x \\
 & & yx
 \end{array}$$

Summarizing, an adjunction in $Q_*\mathcal{K}$ is given by 1-cells $x : A \rightarrow B$ and $y : B \rightarrow A$ together with 2-cells $\varphi : xy \rightarrow 1_B$ and $\psi : 1_A \rightarrow yx$ in \mathcal{K} ; rendering commutative these two diagrams. This structure is discussed in [24] under the name ‘regular adjunction context’. The corresponding demimonad is (A, yx) with the associative multiplication $\bar{y}\varphi\bar{x}$ and demiunit ψ .

2.2. Algebras for monads. A monad (A, t) in a 2-category \mathcal{K} induces a monad $\mathcal{K}(B, t)$ on the category $\mathcal{K}(B, A)$ for any object B of \mathcal{K} . We may consider its Eilenberg-Moore algebras; i.e. the actions of (A, t) on morphisms $B \rightarrow A$. We can now define demiactions of our demimonads as ordinary actions in $Q_*\mathcal{K}$ of the monads in $Q_*\mathcal{K}$. Even if we start with an actual monad in \mathcal{K} , this gives something new.

In view of Proposition 2.2, a demimonad in a 2-category \mathcal{K} is given by a 1-cell $t : A \rightarrow A$, an associative multiplication $\mu : t^2 \rightarrow t$ and a 2-cell $\eta : A \rightarrow t$ subject to the conditions in Proposition 2.2. For the idempotent 2-cell $\mu.t\eta = \mu.\eta t : t \rightarrow t$ we write μ_1 .

Proposition 2.4. *A demi-action of a demimonad (A, t) on a morphism $a : B \rightarrow A$ consists of a 2-cell $\alpha : ta \rightarrow a$ satisfying the associative law $\alpha.t\alpha = \alpha.\mu a$ as well as $\alpha.\mu_1 a = \alpha$; when t is a monad, then $\mu_1 = 1$ and the second condition is automatic.*

PROOF: An action in $Q_*\mathcal{K}$ consists of a morphism $a : B \rightarrow A$ equipped with an idempotent $\bar{a} : a \rightarrow a$, and an action $\alpha : (t, \mu_1)(a, \bar{a}) \rightarrow (a, \bar{a})$. In order for α to be a morphism in $Q_*\mathcal{K}$, we need $\bar{a}.\alpha = \alpha = \alpha.\mu_1 a.t\bar{a}$; or equivalently $\bar{a}.\alpha = \alpha = \alpha.\mu_1 a = \alpha.t\bar{a}$. The associative law says $\alpha.t\alpha = \alpha.\mu a$ and the unit law says $\alpha.\eta a = \bar{a}$. Thus \bar{a} can be recovered from α . We must show that any $\alpha : ta \rightarrow a$ satisfying $\alpha.t\alpha = \alpha.\mu a$ and $\alpha.\mu_1 a = \alpha$ satisfies the remaining conditions.

First of all $\alpha.\eta a.\alpha = \alpha.t\alpha.\eta ta = \alpha.\mu a.\eta ta = \alpha.\mu_1 a = \alpha$, and so $\bar{a}\alpha = \alpha$. Furthermore this gives $\bar{a}\bar{a} = \bar{a}.\alpha.\eta a = \alpha.\eta a = \bar{a}$ and \bar{a} is idempotent. Finally $\alpha.t\bar{a} = \alpha.t\alpha.\eta ta = \alpha.\mu a.\eta ta = \alpha.\mu_1 a = \alpha$. \square

Remark 2.5. A morphism $(b, \beta) \rightarrow (c, \gamma)$ of t -(demi)actions is a 2-cell $\varphi : b \rightarrow c$ making the following diagrams commute.

$$\begin{array}{ccc} tb & \xrightarrow{t\varphi} & tc \\ \beta \downarrow & & \downarrow \gamma \\ b & \xrightarrow{\varphi} & c \end{array} \qquad \begin{array}{ccc} b & \xrightarrow{\varphi} & c \\ \beta.\eta b \downarrow & \searrow \varphi & \downarrow \gamma.\eta c \\ b & \xrightarrow{\varphi} & c. \end{array}$$

Commutativity of the diagram on the left is the usual condition for morphisms of t -actions; as for the diagram on the right, the exterior will commute if the diagram on the left does – the new condition is that the two equal paths around the exterior are themselves equal to the diagonal. Of course if β and γ are genuine (unital) actions, then $\beta.\eta b$ and $\gamma.\eta c$ are identities, and this is automatic. It is important to note that the identity morphism on a demi-action (b, β) is given by $\beta.\eta b : b \rightarrow b$; this is the identity only in the case of a genuine action. It follows that a morphism $\varphi : (b, \beta) \rightarrow (c, \gamma)$ of demi-actions can be invertible without $\varphi : b \rightarrow c$ being invertible in \mathcal{K} .

2.3. The Eilenberg-Moore category. A proper monad (A, t) in \mathcal{K} can be regarded as a demimonad; i.e. a monad (A, t) in $Q_*\mathcal{K}$. For any other object B in \mathcal{K} , the induced monad $Q_*\mathcal{K}(B, t)$ on the category $Q_*\mathcal{K}(B, A)$ is equal to the image of the monad $\mathcal{K}(B, t)$ on $\mathcal{K}(B, A)$ under the 2-functor $Q : \mathbf{Cat} \rightarrow \mathbf{Cat}$ in Section 1.2. Hence to give a demi-action of the demimonad (A, t) is equivalently to give an actual algebra of the latter monad on $Q(\mathcal{K}(B, A))$.

We may apply this reasoning to the particular 2-category $\mathcal{K} = \mathbf{Cat}$ and its terminal object $B = 1$. Then for any monad t on a category A , there is a coinciding notion of a demi-action of the demimonad (A, t) (i.e. action of the monad (A, t) in $Q_*\mathbf{Cat}$) and that of an actual algebra of the monad $Qt = (Qt, Q\mu, Q\eta)$ on QA . We call it a t -demialgebra. From Proposition 2.4 we obtain the following explicit description.

Proposition 2.6. *A t -demialgebra for a monad t is the same thing as an object $b \in A$ equipped with a morphism $\beta : tb \rightarrow b$ satisfying the associative law $\beta.t\beta = \beta.\mu b$.* \square

We obtain a category $A^{(t)}$ of t -demialgebras, by taking as morphisms the Qt -morphisms between the corresponding Qt -algebras. This gives an isomorphism of categories $A^{(t)} \cong (QA)^{Qt}$. Explicitly, if (b, β) and (c, γ) are t -demialgebras, a morphism of demialgebras from (b, β) to (c, γ) is a morphism $\varphi : b \rightarrow c$ satisfying $\varphi.\beta = \gamma.t\varphi$ and $\gamma.\eta c.\varphi = \varphi$. As said in Remark 2.5, the identity morphism on a t -demialgebra (b, β) is $\beta.\eta b$.

Example 2.7. A demialgebra for the identity monad on B is a morphism $\beta : b = 1_B(b) \rightarrow b$ satisfying $\beta\beta = \beta$; that is, an idempotent in B . In symbols: $A^{(1)} = QA$.

Proposition 2.8. *A t -demialgebra (b, β) is isomorphic to a t -algebra if and only if the idempotent $\beta.\eta b$ splits.*

PROOF: Let (a, α) be a t -algebra. An isomorphism $(a, \alpha) \cong (b, \beta)$ consists of t -demialgebra maps $\sigma : (a, \alpha) \rightarrow (b, \beta)$ and $\pi : (b, \beta) \rightarrow (a, \alpha)$ satisfying $\pi\sigma = 1$ and $\sigma\pi = \beta.\eta b$. So certainly if (b, β) is isomorphic to a t -algebra then the idempotent splits. Suppose conversely that the idempotent splits, say as $\beta.\eta b = \sigma\pi$, with $\pi\sigma = 1_a$. Then a inherits a unique demialgebra structure $\alpha : ta \rightarrow a$ such that σ and π are both demialgebra morphisms. It remains to check that (a, α) is in fact an algebra.

Since $\pi : (b, \beta) \rightarrow (a, \alpha)$ is a demialgebra morphism, we have $\alpha.\eta a.\pi = \pi$; but $\pi\sigma = 1$ and so $\alpha.\eta a = 1$. \square

Remark 2.9. We saw before that $Q : \mathbf{Cat} \rightarrow \mathbf{Cat}$ does not preserve powers; it also does not preserve Eilenberg-Moore objects, since the canonical comparison $Q(C^t) \rightarrow (QC)^{Q^t}$ is not invertible; indeed this time it is not even an equivalence in general. It will be an equivalence if and only if each t -demialgebra is a retract (in the category of t -demialgebras) of a t -algebra; in particular, this will be the case if idempotents split in C .

In more detail, an object of $Q(C^t)$ consists of a t -algebra (A, a) and an idempotent t -morphism $e : (A, a) \rightarrow (A, a)$. A morphism from (A, a, e) to (B, b, d) consists of a t -morphism $f : (A, a) \rightarrow (B, b)$ satisfying $df = f.e$.

An object of $Q(C)^{Q(t)}$ consists of an object $A \in C$ with a morphism $a' : tA \rightarrow A$ satisfying the associative law $a'.ta' = a'.\mu A$. A morphism from (A, a') to (B, b') consists of a morphism $f : A \rightarrow B$ satisfying $fa' = b'.tf$ and $fa'.\eta A = f$.

The comparison functor K sends (A, a, e) to (A, ea) and a morphism $f : (A, a, e) \rightarrow (B, b, d)$ to $f : (A, ea) \rightarrow (B, db)$. Clearly this is faithful; while given an arbitrary $f : (A, ea) \rightarrow (B, db)$ we have $f = fea.\eta A = fe$ and $f = fea.\eta A = db.tf.\eta A = db.\eta B.f = df$, and so also $fa = fea = db.tf = b.td.tf = b.t(df) = b.tf$, which proves that f is a morphism $(A, a, e) \rightarrow (B, b, d)$ and so that K is also full.

For any object (B, b, d) of $Q(B^t)$, $K(B, b, d) = (B, db)$ is a retract in $(QB)^{Q^t}$ of the t -algebra (B, b) via the epimorphism $d : (B, b) \rightarrow (B, db)$ and its section $d : (B, db) \rightarrow (B, b)$. Hence a t -demialgebra (a, α) will be isomorphic to an object in the image of K if and only if it is a retract of a t -algebra. Thus K will be an equivalence if every t -demialgebra is a retract (in the category of demialgebras) of a t -algebra. It will of course be an equivalence whenever idempotents split in C . In general, however, K needs not be surjective on objects, or even essentially surjective, and so will not be an equivalence of categories.

As an example, consider the category of categories with chosen initial object, and functors preserving the chosen initial object. This has a subcategory B consisting of the finite ordinals $n = \{0 < 1 < \dots < n-1\}$ with $n \geq 2$, and the category \mathcal{J} with a chosen initial object 0 and another initial object 0'; we include all maps except that we only allow functors $n \rightarrow \mathcal{J}$ which are constant at 0. There is an evident monad t which adjoins a top element (except that when applied to \mathcal{J} it first collapses 0 and 0' to a single element 0). Each n has a unique t -algebra structure $\alpha : n+1 \rightarrow n$ which collapses the top two elements of $n+1$. The unique map $2 \rightarrow \mathcal{J}$ makes \mathcal{J} into a demialgebra (but not an algebra). Any map $\varphi : \mathcal{J} \rightarrow n$ must satisfy $\varphi(0) = 0$; but to be a demialgebra map it would also need to satisfy $\varphi(0) = n-1$ which is clearly impossible for $n \geq 2$. Thus there is *no* demialgebra map from \mathcal{J} to a t -algebra, and so certainly \mathcal{J} is not a retract of a t -algebra.

2.4. Monoids as algebras of the free monoid monad. We shall now work through in some detail a not entirely trivial example. Let B be a monoidal category with countable coproducts over which the tensor product distributes. We write, for convenience, as if B were strict. Then free monoids in B can be constructed via the usual geometric series $tb = \sum_n b^n$, where b^n denotes the n -fold tensor power of an object b . Then t becomes a monad on B , cf. [16, p. 172, Theorem 2]. A t -demialgebra consists of an object (b, ρ) of QB equipped with an action β of Qt . To give a map $\beta : tb \rightarrow b$ is to give a map $\beta_n : b^n \rightarrow b$ for each n . The unit law $\beta.\eta b = \rho$ says that $\beta_1 = \rho$. The fact that β is a morphism $(tb, t\rho) \rightarrow (b, \rho)$ in QB amounts to the equations $\rho\beta_n = \beta_n$ and $\beta_n\rho^n = \beta_n$. The associativity constraint can be written as commutativity of

$$(1) \quad \begin{array}{ccc} b^{\sum_{k=1}^n m_k} & \xrightarrow{\beta_{\sum_{k=1}^n m_k}} & b \\ \otimes_{k=1}^n \beta_{m_k} \downarrow & & \parallel \\ b^n & \xrightarrow{\beta_n} & b \end{array}$$

for all natural numbers n, m_1, \dots, m_n . Putting $(n = 2, m_1 = 0, m_2 = 1)$ and $(n = 2, m_1 = 1, m_2 = 0)$, (1) reduces to the conditions

$$(2) \quad \beta_2(\beta_0 \otimes b) = \beta_1 = \beta_2(b \otimes \beta_0).$$

Evaluating (1) at $(n = 2, m_1 = 1, m_2 = 2)$ and $(n = 2, m_1 = 2, m_2 = 1)$ gives the associativity of β_2 . Finally, taking $(n = 2, m_1 = p - 1, m_2 = 1)$ for any positive integer p , and iterating the resulting relation, we obtain

$$\beta_p = \beta_2(\beta_{p-1} \otimes b) = \dots = \beta_2(\beta_2 \otimes b) \dots (\beta_2 \otimes b^{p-2}).$$

Together with the associativity of β_2 , this identity implies commutativity of (1) for any values of n and m_1, \dots, m_n . Putting all that together, we see that the entire structure consists of (i) an associative multiplication $\beta_2 : b \otimes b \rightarrow b$, (ii) an idempotent $\beta_1 : b \rightarrow b$ satisfying $\beta_1 \beta_2 = \beta_2 = \beta_2(\beta_1 \otimes b) = \beta_2(1 \otimes \beta_1)$, and (iii) a map $\beta_0 : i \rightarrow b$ satisfying $\beta_1 \beta_0 = \beta_0$ and (2). Of course β_1 is determined by β_2 and β_0 . A morphism of t -demialgebras from (b, β_2, β_0) to (c, γ_2, γ_0) is a morphism $\varphi : b \rightarrow c$ commuting with the structure maps and satisfying $\varphi \beta_1 = \varphi$.

Comparing the above description of the category of t -demialgebras and the category of demimonoids in Section 2.1, we obtain the following.

Theorem 2.10. *Suppose that B is a monoidal category with countable coproducts, and that the monoidal structure distributes over the coproducts. Then the category of demimonoids in B is just the category $B^{(t)}$ of t -demialgebras.*

This extends the well-known isomorphism between the category of monoids in B and the category B^t of algebras for the free monoid monad t .

2.5. The 2-category of monads. Given a monad t on an object A of a 2-category \mathcal{K} , an action of t on a morphism $a : B \rightarrow A$ is a special case of the notion of morphism of monad. In fact, for every object $B \in \mathcal{K}$, there is an identity monad 1 on B , and to give a morphism $a : B \rightarrow A$ and an action of t on a is equivalently to give a monad morphism from $(B, 1)$ to (A, t) . Similarly, one has 2-cells between monad morphisms, and indeed a whole 2-category $\mathbf{Mnd}(\mathcal{K})$ of monads in \mathcal{K} . This was introduced and studied in [19]; there is also a variant $\mathbf{EM}(\mathcal{K})$ with a different notion of 2-cell which was proposed in [14]. In the subsequent sections we shall develop weak analogues of these.

3. LAX FUNCTORS, LAX NATURAL TRANSFORMATIONS AND MODIFICATIONS

For any 2-categories \mathcal{C} and \mathcal{K} , there is a bicategory of lax functors $\mathcal{C} \rightarrow \mathcal{K}$, lax natural transformations between them, and their modifications. Its ‘weak’ analog is obtained below by replacing the target 2-category \mathcal{K} by its local Cauchy completion $Q_*\mathcal{K}$.

3.1. Lax functors. Let \mathcal{C} and \mathcal{K} be 2-categories. The notion of lax functor from \mathcal{C} to \mathcal{K} was introduced in [3] under the name “morphism of bicategories”, in the more general context where \mathcal{C} and \mathcal{K} were bicategories. A lax functor F differs from a 2-functor by the property that it preserves horizontal composition and identity 1-cells only up-to natural transformations $\mu : F(-).F(-) \rightarrow F(-.)$ and $\eta : 1_{F(-)} \rightarrow F(1_{(-)})$, respectively, which obey coherence conditions called associativity and unitality.

Example 3.1. Consider the case $\mathcal{C} = 1$, where 1 is the 2-category with a single object $*$ and trivial hom-category $1(*, *) = 1$. To give the object-part of a lax functor $1 \rightarrow \mathcal{K}$ is to give an object $A \in \mathcal{K}$; to give the functors between hom-categories is to give a functor $1 \rightarrow \mathcal{K}(A, A)$; that is, to give a 1-cell $t : A \rightarrow A$ in \mathcal{K} . There is only one component of μ to worry about: it is a 2-cell $\mu : t^2 \rightarrow t$. Similarly the only component of η is a 2-cell $\eta : 1 \rightarrow t$. The associativity and unit conditions say precisely that (t, μ, η) is a monad.

As was already anticipated in the previous section, since a monad in \mathcal{K} is a lax functor $1 \rightarrow \mathcal{K}$, a demimonad in \mathcal{K} is a lax functor $1 \rightarrow Q_*\mathcal{K}$. We therefore define, more generally, a *lax demifunctor* from \mathcal{C} to \mathcal{K} to be a lax functor from \mathcal{C} to $Q_*\mathcal{K}$.

These lax demifunctors will be of less importance themselves than their morphisms, introduced below. Nonetheless we shall take the trouble to spell out the structure in more direct terms. First of all, for each

object $C \in \mathcal{C}$, an object $FC \in \mathcal{K}$ is given. For all objects $C, D \in \mathcal{C}$, a functor $\mathcal{C}(C, D) \rightarrow Q_*\mathcal{K}(FC, FD)$ is given; that is, a functor $\mathcal{C}(C, D) \rightarrow Q(\mathcal{K}(FC, FD))$, or equivalently, a semifunctor $\mathcal{C}(C, D) \rightarrow \mathcal{K}(FC, FD)$. As usual, an identity 2-cell 1_f in \mathcal{C} will be sent to an idempotent 2-cell $F1_f : Ff \rightarrow Ff$ in \mathcal{K} . For 1-cells $f : C \rightarrow D$ and $g : D \rightarrow E$ in \mathcal{C} , we have a 2-cell $\mu : Fg.Ff \rightarrow F(gf)$ in \mathcal{K} . It is natural in f and g , and satisfies the usual associativity condition, expressed by commutativity of the first diagram below, as well as a normalization condition which states that the two triangles in the second diagram commute, for all arrows $e : B \rightarrow C$, $f : C \rightarrow D$, $g : D \rightarrow E$ in \mathcal{C} :

$$\begin{array}{ccc} Fg.Ff.Fe \xrightarrow{\mu_{g,f} \cdot 1_{Fe}} F(gf).Fe & & Fg.Ff \xrightarrow{\mu} F(gf) \\ \downarrow 1_{Fg} \cdot \mu_{f,e} & & \downarrow \mu_{g,f} \cdot e \\ Fg.F(fe) \xrightarrow{\mu_{g,fe}} F(gfe) & & \begin{array}{ccc} F1_g.F1_f & \searrow \mu & \downarrow F1_{gf} \\ Fg.Ff & \xrightarrow{\mu} & F(gf) \end{array} \end{array}$$

Similarly, for each $C \in \mathcal{C}$ there is a 2-cell $\eta : 1_{FC} \rightarrow F1_C$, but the unit conditions now say that the composites

$$Ff \xlongequal{\quad} Ff.1_{FC} \xrightarrow{1_{Ff} \cdot \eta} Ff.F1_C \xrightarrow{\mu_{f,1_C}} Ff \quad \text{and} \quad Ff \xlongequal{\quad} 1_{FD}.Ff \xrightarrow{\eta \cdot 1_{Ff}} F1_D.Ff \xrightarrow{\mu_{1_D,f}} Ff$$

are equal to the idempotent $F1_f$; this time the normalization condition states that the composite

$$1_{FC} \xrightarrow{\eta} F1_C \xrightarrow{F1_{1_C}} F1_C$$

is just η .

3.2. Lax natural transformations. Once again, any lax functor $F : \mathcal{C} \rightarrow \mathcal{K}$ determines a lax demifunctor $q_*F : \mathcal{C} \rightarrow Q_*\mathcal{K}$ with which it is identified. For such a lax demifunctor, $F1_f = 1_{Ff}$ for any 1-cell f in \mathcal{C} . Even for such lax functors $F, G : \mathcal{C} \rightarrow \mathcal{K}$, however, we obtain a new type of morphism, namely the lax natural transformations $q_*F \rightarrow q_*G$.

What then is a lax natural transformation between lax functors $F, G : \mathcal{C} \rightarrow Q_*\mathcal{K}$? For each $C \in \mathcal{C}$ we should give a 1-cell $FC \rightarrow GC$ in $Q_*\mathcal{K}$; in other words, a 1-cell $xC : FC \rightarrow GC$ along with an idempotent 2-cell $\bar{x}C : xC \rightarrow xC$. Next, for each 1-cell $f : C \rightarrow D$ in \mathcal{C} , we should give a 2-cell $(Gf, G1_f)(xC, \bar{x}C) \rightarrow (xD, \bar{x}D)(Ff, F1_f)$ in $Q_*\mathcal{K}$; that is, a 2-cell

$$\begin{array}{ccc} FC & \xrightarrow{xC} & GC \\ Ff \downarrow & \Downarrow xf & \downarrow Gf \\ FD & \xrightarrow{x_D} & GD \end{array}$$

for which the two composites

$$Gf.xC \xrightarrow{xf} xD.Ff \xrightarrow{\bar{x}D.F1_f} xD.Ff \quad \text{and} \quad Gf.xC \xrightarrow{G1_f \cdot \bar{x}C} Gf.xC \xrightarrow{xf} xD.Ff$$

are both just xf . This x obeys the same naturality condition and the same compatibility with composition as a usual lax natural transformation between lax functors $\mathcal{C} \rightarrow \mathcal{K}$: the same diagrams

$$\begin{array}{ccc} \text{(LN0)} & Gf.xC \xrightarrow{xf} xD.Ff & \text{(LN1)} & Gg.Gf.xC \xrightarrow{1_{Gg} \cdot xf} Gg.xD.Ff \xrightarrow{xg \cdot 1_{Ff}} xE.Fg.Ff \\ G\alpha \cdot 1_{xC} \downarrow & \downarrow 1_{xD} \cdot F\alpha & \mu_{g,f}^G \cdot 1_{xC} \downarrow & \downarrow 1_{xE} \cdot \mu_{g,f}^F \\ Gf'.xC \xrightarrow{xf'} xD.Ff' & & G(gf).xC \xrightarrow{x(gf)} xE.F(gf) \end{array}$$

commute for all 1-cells $f, f' : C \rightarrow D$, $g : D \rightarrow E$ and 2-cells $\alpha : f \rightarrow f'$. The third condition, expressing compatibility with identities, is changed because the identity 2-cell in $Q_*\mathcal{K}$ on $(xC, \bar{x}C)$ is $\bar{x}C$. Thus the

new condition becomes commutativity of

(DLN2)

$$\begin{array}{ccc} 1_{GC}.xC & \xlongequal{\quad} & xC.1_{FC} \\ \eta^G.\overline{x}C \downarrow & & \downarrow \overline{x}C.\eta^F \\ G1_C.xC & \xrightarrow{x1_C} & xC.F1_C. \end{array}$$

For lax functors $F, G : \mathcal{C} \rightarrow \mathcal{K}$, we may consider lax (or, alternatively, pseudo) natural transformations $q_*F \rightarrow q_*G$. (For an explicit description, substitute in the above diagrams by 1_{Ff} and 1_{Gf} the idempotents $F1_f$ and $G1_f$, respectively, for any 1-cell f .) We call such a structure a *lax (or, alternatively, pseudo) demitransformation* from F to G .

This simplifies somewhat if F and G are in fact 2-functors:

Proposition 3.2. *Let $F, G : \mathcal{C} \rightarrow \mathcal{K}$ be 2-functors. A lax demitransformation from F to G consists of a morphism $xC : FC \rightarrow GC$ in \mathcal{K} for each $C \in \mathcal{C}$ equipped with 2-cells $xf : Gf.xC \rightarrow xD.Ff$ satisfying conditions (LN0) and (LN1) only.*

PROOF: We shall show that $\overline{x}C$ is just $x1_C$, and that all conditions involving it are then automatically satisfied. First of all, by compatibility with composition (LN1) $x1_C$ is clearly idempotent. By (DLN2) we have $x1_C.\overline{x}C = \overline{x}C$, while the fact that $x1_C$ is a 2-cell in $Q_*\mathcal{K}$ gives $x1_C.\overline{x}C = x1_C$. Thus $\overline{x}C$ is necessarily just $x1_C$.

So now we define $\overline{x}C$ to be $x1_C$. Clearly (LN0), (LN1), and (DLN2) hold; we need only check that the composites

$$Gf.xC \xrightarrow{xf} xD.Ff \xrightarrow{x1_D.1_{Ff}} xD.Ff \quad \text{and} \quad Gf.xC \xrightarrow{1_{Gf}.x1_C} Gf.xC \xrightarrow{xf} xD.Ff$$

are both xf ; these are both instances of compatibility with composition (LN1). \square

3.3. Modifications. Finally we consider morphisms between lax natural transformations, called *modifications*. In the case of lax demitransformations $(x, \overline{x}), (y, \overline{y}) : F \rightarrow G$, we retain the same word: a modification from (x, \overline{x}) to (y, \overline{y}) consists of a 2-cell $\xi C : xC \rightarrow yC$ in \mathcal{K} for each $C \in \mathcal{C}$, subject to the usual condition expressed by commutativity of (M) as well as the extra condition represented by (DM):

(M)

$$\begin{array}{ccc} Gf.xC & \xrightarrow{xf} & xD.Ff \\ 1_{Gf}.\xi C \downarrow & & \downarrow \xi D.1_{Ff} \\ Gf.yC & \xrightarrow{yf} & yD.Ff \end{array}$$

(DM)

$$\begin{array}{ccc} xC & \xrightarrow{\xi C} & yC \\ \overline{x}C \downarrow & \searrow \xi C & \downarrow \overline{y}C \\ xC & \xrightarrow{\xi C} & yC \end{array}$$

Of course in the case of lax natural transformations, $\overline{x}C$ and $\overline{y}C$ are identities and so commutativity of (DM) is automatic.

4. LIMITS

We now turn to the notion of limit within our “weak world”. Because of the well-established sense of “weak limit”, referred to in the introduction, we henceforth drop completely the epithet “weak”, and speak only of *demilimits*.

For an ordinary functor $S : \mathcal{C} \rightarrow \mathcal{K}$, the limit of S , if it exists, is defined as the representing object of the functor from \mathcal{C} to **Set** sending $C \in \mathcal{C}$ to the set of cones under S with vertex C . Such a cone is of course just a natural transformation from the constant functor ΔC at C to S . Thus the notion of limit depends, among other things, on the notion of naturality. In the 2-categorical context, there is the possibility to replace naturality by lax naturality, giving rise to a notion of lax limit [20]; but in light of the previous section we could instead consider (lax) deminaturality and so obtain a notion of demilimit. (For more on bilimits and lax limits see [20] or [11].)

For (small) 2-categories \mathcal{C} and \mathcal{K} we write $[\mathcal{C}, \mathcal{K}]$ for the usual 2-category of 2-functors from \mathcal{C} to \mathcal{K} , with 2-natural transformations as 1-cells and modifications of 2-cells. Clearly, there is a natural bijection

between 2-functors $\mathcal{K} \rightarrow [\mathcal{C}, \mathcal{K}]$ and $\mathcal{K} \times \mathcal{C} \rightarrow \mathcal{K}$. We write $\text{Ps}(\mathcal{C}, \mathcal{K})_{\text{lax}}$ for the bicategory of lax functors from \mathcal{C} to \mathcal{K} , with pseudonatural transformations as 1-cells, and modifications as 2-cells. We denote by J the fully faithful inclusion $\mathbf{Cat}_{\text{cc}} \rightarrow \mathbf{Cat}$, of the full sub-2-category of \mathbf{Cat} consisting of the Cauchy complete categories.

4.1. Weighted bilimits. Let \mathcal{C} be a 2-category and $S : \mathcal{C} \rightarrow \mathcal{K}$ be a lax demifunctor (of course this includes the case of an ordinary lax functor, or indeed of a 2-functor); let $F : \mathcal{C} \rightarrow \mathbf{Cat}_{\text{cc}}$ be a 2-functor. The *demilimit of S weighted by F* is defined to be the JF -weighted bilimit of the lax functor $S : \mathcal{C} \rightarrow Q_*\mathcal{K}$. That is, an object $\text{dl}(F, S)$ of $Q_*\mathcal{K}$ (i.e. of \mathcal{K}) equipped with a pseudonatural equivalence

$$Q_*\mathcal{K}(-, \text{dl}(F, S)) \simeq \text{Ps}(\mathcal{C}, \mathbf{Cat})_{\text{lax}}(JF, JQ_*\mathcal{K}(-, S)),$$

equivalently,

$$Q_*\mathcal{K}(-, \text{dl}(F, S)) \simeq \text{Ps}(\mathcal{C}, \mathbf{Cat}_{\text{cc}})_{\text{lax}}(F, Q_*\mathcal{K}(-, S)).$$

If also the domain 2-category \mathcal{C} is locally Cauchy complete, then this notion of demilimit of a lax demifunctor $\mathcal{C} \rightarrow \mathcal{K}$ coincides with the bilimit of the respective lax functor $\mathcal{C} \rightarrow Q_*\mathcal{K}$ in the \mathbf{Cat}_{cc} enriched sense.

Similarly we have the *lax demilimit* $\text{dll}(F, S)$ defined by

$$Q_*\mathcal{K}(-, \text{dll}(F, S)) \simeq \text{Lax}(\mathcal{C}, \mathbf{Cat}_{\text{cc}})_{\text{lax}}(F, Q_*\mathcal{K}(-, S)).$$

Note that $\text{dll}(F, S)$ can be constructed as $\text{dl}(F', S)$ in terms of an appropriate weight F' .

A *demicolimit* in \mathcal{K} is of course just a demilimit notion in \mathcal{K}^{op} .

Example 4.1. Let $T : \mathcal{C} \rightarrow \mathbf{Cat}_{\text{cc}}$ be the 2-functor constant on the terminal category 1. The demilimit $\text{dl}(T, S)$ of a lax demifunctor $S : \mathcal{C} \rightarrow \mathcal{K}$ is defined by the pseudonatural equivalence

$$Q_*\mathcal{K}(-, \text{dl}(T, S)) \simeq \text{Ps}(\mathcal{C}, \mathbf{Cat})_{\text{lax}}(JT, JQ_*\mathcal{K}(-, S)) \simeq \text{Ps}(\mathcal{C}, Q_*\mathcal{K})_{\text{lax}}(\Delta(-), S),$$

cf. [20, Section 4], where $\Delta : \mathcal{K} \rightarrow [\mathcal{C}, \mathcal{K}] \rightarrow \text{Ps}(\mathcal{C}, \mathcal{K})_{\text{lax}}$ denotes the diagonal 2-functor (with the first arrow corresponding to the first projection $\mathcal{K} \times \mathcal{C} \rightarrow \mathcal{K}$ and the second one being the obvious inclusion). Thus for this particular weight T , the demilimit $\text{dl}(T, S)$ is directly related to the bicategory of lax demifunctors, demitransformations and modifications in Section 3.

Our primary focus will be the case of Eilenberg-Moore objects, to which we turn in the following section.

4.2. Eilenberg-Moore objects. We have already seen the notion of Eilenberg-Moore object: for a monad t on a category B we write B^t for the category of t -algebras; for a monad t on an object B of a 2-category \mathcal{K} , we write B^t for the representing object in a representation

$$\mathcal{K}(A, B^t) \cong \mathcal{K}(A, B)^{\mathcal{K}(A, t)}$$

where $\mathcal{K}(A, t)$ is the induced monad on the hom-category $\mathcal{K}(A, B)$. As a first generalization, we do not ask for an isomorphism, but just a pseudonatural equivalence

$$\mathcal{K}(A, B^t) \simeq \mathcal{K}(A, B)^{\mathcal{K}(A, t)}$$

and we then call B^t a *bicategorical EM-object*.

This fits into the framework of weighted limits of the previous section, more specifically the situation in Example 4.1. We take \mathcal{C} to be the terminal 2-category, then a monad (B, t) in \mathcal{K} is simply a lax functor $S : \mathcal{C} \rightarrow \mathcal{K}$. We take the weight $T : \mathcal{C} \rightarrow \mathbf{Cat}$ to be the 2-functor constant at the terminal category. Then a lax natural transformation from T to $\mathcal{K}(A, S)$ consists of a single component $b : A \rightarrow B$, with lax naturality constraint in the form of a 2-cell $\beta : tb \rightarrow b$, with the conditions stating (b, β) is a t -algebra. Thus the bicategorical Eilenberg-Moore object of t is just the lax bilimit of the corresponding lax functor $S : 1 \rightarrow \mathcal{K}$ (weighted by $T : 1 \rightarrow \mathbf{Cat}$).

We now look in detail at the “demi” version. A demimonad in \mathcal{K} is a monad in $Q_*\mathcal{K}$. For such a monad, a bicategorical EM-object amounts to a representing object for $Q_*\mathcal{K}(A, B)^{Q_*\mathcal{K}(A, t)}$, as a 2-functor of $A \in Q_*\mathcal{K}$. Then we seek a pseudonatural equivalence

$$(3) \quad Q_*\mathcal{K}(A, B^{(t)}) \simeq Q_*\mathcal{K}(A, B)^{Q_*\mathcal{K}(A, t)}.$$

The right hand side is the category of demiactions of t (cf. Proposition 2.4). The universal property guarantees a morphism $u : B^{(t)} \rightarrow B$ with a demiaction, $\psi : tu \rightarrow u$, such that for any demiaction $(a : A \rightarrow B, \alpha : ta \rightarrow a)$, there exists a morphism $(x, \bar{x}) : A \rightarrow B^{(t)}$ and an isomorphism of demialgebras $\xi : (ux, \psi\bar{x}) \cong (a, \alpha)$.

This becomes particularly simple in the case where t is actually a monad in \mathcal{K} . Then the right hand side becomes just $\mathcal{K}(A, B)^{(\mathcal{K}(A, t))}$, and we seek a pseudonatural equivalence

$$Q_*\mathcal{K}(A, B^{(t)}) \simeq \mathcal{K}(A, B)^{(\mathcal{K}(A, t))}.$$

We can make this more explicit as follows. There is a morphism $u : B^{(t)} \rightarrow B$, equipped with an action $\psi : tu \rightarrow u$ satisfying the associative law $\psi.t\psi = \psi.\mu u$, but not required to satisfy the unit law. From the requirement that it induces an equivalence, it has the following universal properties. By essential surjectivity on objects, for any morphism $a : A \rightarrow B$ and any demiaction $\alpha : ta \rightarrow a$, there exists a morphism $(x, \bar{x}) : A \rightarrow B^{(t)}$ in $Q_*\mathcal{K}$ and an isomorphism $\xi : (u, \psi_1 := \psi.\eta u)(x, \bar{x}) \cong (a, \alpha.\eta a)$ in $Q_*\mathcal{K}$ for which the diagram

$$\begin{array}{ccc} tux & \xrightarrow{t\xi} & ta \\ \psi x \downarrow & & \downarrow \alpha \\ ux & \xrightarrow{\xi} & a \end{array}$$

commutes. Furthermore, there is a 2-dimensional aspect coming from the fact that (u, ψ) induces a fully faithful functor. Let $(x, \bar{x}), (y, \bar{y}) : A \rightarrow B^{(t)}$ be given. For any $\zeta : ux \rightarrow uy$ for which the diagrams

$$\begin{array}{ccc} tux & \xrightarrow{t\zeta} & tuy \\ \psi x \downarrow & & \downarrow \psi y \\ ux & \xrightarrow{\zeta} & uy \end{array} \quad \begin{array}{ccc} ux & \xrightarrow{\zeta} & uy \\ \psi_1 x \downarrow & \searrow \zeta & \downarrow \psi_1 y \\ ux & & uy \\ u\bar{x} \downarrow & & \downarrow u\bar{y} \\ ux & \xrightarrow{\zeta} & uy \end{array}$$

commute, there is a unique 2-cell $\zeta' : x \rightarrow y$ with $\zeta'\bar{x} = \zeta' = \bar{y}\zeta'$ and $u\zeta'.\psi_1 x = \zeta$.

For any demimonad (B, t) in a 2-category \mathcal{K} , the image of the object $((t, \mu_1), \mu)$ of $Q_*\mathcal{K}(B, B)^{Q_*\mathcal{K}(B, t)}$ under the isomorphism (3) provides a left adjoint $(f, \bar{f}) : B \rightarrow B^{(t)}$ of the 1-cell (u, ψ_1) in $Q_*\mathcal{K}$ above. By virtue of the universal property we have seen, the corresponding monad in $Q_*\mathcal{K}$ is isomorphic to (B, t) . This means the existence of 2-cells $\chi : uf \rightarrow t$ and $\chi' : t \rightarrow uf$ obeying the normalization conditions

$$\begin{array}{ccc} uf & \xrightarrow{\chi} & t \\ \psi_1 \bar{f} \downarrow & \searrow \chi & \downarrow \mu_1 \\ uf & \xrightarrow{\chi} & t \end{array} \quad \begin{array}{ccc} t & \xrightarrow{\chi'} & uf \\ \mu_1 \downarrow & \searrow \chi' & \downarrow \psi_1 \bar{f} \\ t & \xrightarrow{\chi'} & uf \end{array} \quad \begin{array}{ccc} uf & \xrightarrow{\chi} & t \\ \psi_1 \bar{f} \downarrow & & \downarrow \chi' \\ & & uf \end{array} \quad \begin{array}{ccc} t & \xrightarrow{\chi'} & uf \\ \mu_1 \downarrow & & \downarrow \chi \\ & & t \end{array}$$

and the ‘ t -linearity’ condition

$$\begin{array}{ccc} tuf & \xrightarrow{t\chi} & t^2 \\ \psi f \downarrow & & \downarrow \mu \\ uf & \xrightarrow{\chi} & t \end{array} \quad \text{equivalently,} \quad \begin{array}{ccc} t^2 & \xrightarrow{t\chi'} & tuf \\ \mu \downarrow & & \downarrow \psi f \\ t & \xrightarrow{\chi'} & uf \end{array}$$

The counit $(f, \bar{f})(u, \psi_1) \rightarrow 1$ takes the form of a map $\epsilon : fu \rightarrow 1$ for which the diagrams

$$\begin{array}{ccc}
 ufu & \xrightarrow{\chi^u} & tu \\
 \psi_1 fu \downarrow & & \downarrow \psi \\
 ufu & \xrightarrow{u\epsilon} & u
 \end{array}
 \qquad
 \begin{array}{ccc}
 fu & \xrightarrow{\bar{f}\psi_1} & fu \\
 \epsilon \searrow & & \downarrow \epsilon \\
 & & 1
 \end{array}$$

commute.

As usual [19] there are various dualities. We write \mathcal{K}^{co} for the 2-category obtained from \mathcal{K} by formally reversing the direction of the 2-cells, but leaving the 1-cells unchanged. A monad in \mathcal{K}^{co} is a comonad in \mathcal{K} , and its demi-EM-object is just called the demi-EM-object of the comonad. We write \mathcal{K}^{op} for the 2-category obtained from \mathcal{K} by formally reversing the direction of the 1-cells, but leaving the 2-cells unchanged. A monad in \mathcal{K}^{op} is still just a monad, but the demi-EM-object is now a *colimit* in \mathcal{K} , called the demi-KL-object (KL for Kleisli). Finally we write $\mathcal{K}^{\text{coop}}$ for the 2-category obtained from \mathcal{K} by reversing both the 1-cells and the 2-cells. A monad in $\mathcal{K}^{\text{coop}}$ is a comonad in \mathcal{K} ; its demi-EM-object is called the demi-KL-object of the comonad.

5. FREE COMPLETIONS

For a small category \mathcal{C} , the presheaf category $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the free completion of \mathcal{C} under colimits. More generally, the free completion of \mathcal{C} under some class of colimits is the closure of the representables in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ under those colimits. For example, the free completion of \mathcal{C} under coproducts is the full subcategory of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ consisting of those objects which are coproducts of representables.

Furthermore, this remains true in the enriched context: if \mathcal{V} is a complete and cocomplete symmetric monoidal closed category, and \mathcal{C} is a small \mathcal{V} -category, the presheaf category $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ is the free completion of \mathcal{C} under colimits; and the closure of the representables in $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ under a given class of colimits is the free completion under those colimits [12]. In particular this can be done in the case $\mathcal{V} = \mathbf{Cat}$ of 2-categories, leading to a description of the free completion of a 2-category under Kleisli objects, or dually under Eilenberg-Moore objects: this was the basis for the main construction in [14]. A potentially tricky aspect of these free completions, is that one does not know how many steps may be involved in forming the closure of the representables under some class of colimits: after each step there will be new diagrams of which to form the colimit, and this process could potentially continue transfinitely. In the case of Kleisli objects, however, it terminates after a single step; this is basically because a functor $f : C \rightarrow D$ exhibits D as a Kleisli object if and only if it is bijective on objects and has a right adjoint, and such functors are closed under composition.

We shall now consider “demi” versions of these ideas. In fact we treat in detail only the case of completions under demi-KL-objects, but many other classes of demicolimits can be handled in similar fashion. In Section 4 we defined demi-KL-objects as bilimits with respect to a certain \mathbf{Cat}_{cc} -valued weight. When taking the free completion under these bilimits, the key idea is to work not with categories enriched in \mathbf{Cat} (2-categories) but rather with categories enriched in \mathbf{Cat}_{cc} , the full 2-subcategory of \mathbf{Cat} consisting of the Cauchy complete categories.

The category \mathbf{Cat}_{cc} is Cartesian closed, so there is no problem enriching over it: a \mathbf{Cat}_{cc} -category is precisely a \mathbf{Cat} -category in which idempotent 2-cells split. The problem is that, as a category, \mathbf{Cat}_{cc} is neither complete nor cocomplete, and so we cannot apply the Kelly theorem. One way around this is to note that although \mathbf{Cat}_{cc} is not complete or cocomplete as a category, it is bicategorically complete and cocomplete (as a 2-category). We can therefore use a bicategorical variant of the Kelly theorem, which we prove in the Appendix. For a small \mathbf{Cat}_{cc} -category \mathcal{C} , we write $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\text{cc}})$ for the 2-category (in fact \mathbf{Cat}_{cc} -category) of pseudofunctors, pseudonatural transformations, and modifications from \mathcal{C}^{op} to \mathbf{Cat}_{cc} . Then $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\text{cc}})$ is the free completion of the \mathbf{Cat}_{cc} -category \mathcal{C} under bicategorical \mathbf{Cat}_{cc} -colimits, while the free completion under bicategorical KL-objects is the closure under such of the representables in $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\text{cc}})$.

Once again, this process is potentially transfinite, but just as in the case of completion under ordinary Kleisli objects, considered in [14], the process terminates after a single step. The key observation here is the following lemma. Before stating it, it is useful to define a functor $f : A \rightarrow B$ to be *quasi-surjective on objects* if every object $b \in B$ is a retract of some fa with $a \in A$. Such functors are clearly closed under composition.

Lemma 5.1. *A morphism $f : A \rightarrow B$ is of bicategorical Kleisli type in $\mathbf{Cat}_{\mathbf{cc}}$ if and only if it has a right adjoint and is quasi-surjective on objects.*

PROOF: Let A be a Cauchy complete category, t a monad on A , and $f_t : A \rightarrow A_t$ its Kleisli category; this is also a bicategorical Kleisli object in \mathbf{Cat} . Since $Q : \mathbf{Cat} \rightarrow \mathbf{Cat}_{\mathbf{cc}}$ is left biadjoint to the inclusion $\mathbf{Cat}_{\mathbf{cc}} \rightarrow \mathbf{Cat}$, it preserves bicategorical Kleisli objects, and so the bicategorical Kleisli object in $\mathbf{Cat}_{\mathbf{cc}}$ of t is the composite

$$A \xrightarrow{f_t} A_t \xrightarrow{q} QA_t.$$

Now A is Cauchy complete, and limits in the Eilenberg-Moore category A^t can be formed as in A , so A^t is also Cauchy complete. The canonical comparison $A_t \rightarrow A^t$ is fully faithful, and so QA_t can be constructed, up to equivalence, as the full subcategory of A^t consisting of all retracts of free algebras. It follows that the composite

$$QA_t \xrightarrow{j} A^t \xrightarrow{u^t} A$$

where j is the inclusion, is right adjoint to qf_t . On the other hand qf_t is clearly quasi-surjective on objects. This proves one half of the characterization.

Suppose conversely that a functor $f : A \rightarrow B$ in $\mathbf{Cat}_{\mathbf{cc}}$ has a right adjoint $f \dashv u$ and is quasi-surjective on objects. We may form the induced monad t on A ; then the Kleisli category A_t can be constructed by factorizing f as an identity on object functor $f : A \rightarrow A_t$ followed by a fully faithful one $q : A_t \rightarrow B$. Now B is Cauchy complete, and contains A_t as a full subcategory, while every object of B is a retract of one in A_t ; it follows that B is equivalent to the Cauchy completion QA_t of A_t . \square

Since bicategorical colimits in $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\mathbf{cc}})$ are constructed pointwise, we get a corresponding characterization of morphisms in $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\mathbf{cc}})$ which are of bicategorical Kleisli type: the pseudonatural transformations which pointwise have right adjoints and are quas surjective on objects. Once again, such morphisms are clearly closed under composition. It is this last fact which means that we need only consider bicategorical Kleisli objects of monads on representables.

At this point we may simply write down an explicit description of the free $\mathbf{Cat}_{\mathbf{cc}}$ -completion $\text{KL}_{\text{dm}}(\mathcal{K})$ of a small $\mathbf{Cat}_{\mathbf{cc}}$ -category \mathcal{C} under bicategorical Kleisli objects. An object is a monad (A, t) in \mathcal{C} (with multiplication μ and unit η understood). This generates a monad in $\text{Hom}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_{\mathbf{cc}})$ on the representable $\mathcal{C}(-, A)$. The Kleisli object is formed by first constructing the pointwise Kleisli object in \mathbf{Cat} , then applying Q , to get

$$\mathcal{C}(X, A) \xrightarrow{F} \mathcal{C}(X, A)_{\mathcal{C}(X, t)} \xrightarrow{q} Q(\mathcal{C}(X, A)_{\mathcal{C}(X, t)}).$$

A morphism from (A, t) to (B, s) should be a pseudonatural transformation

$$Q(\mathcal{C}(X, A)_{\mathcal{C}(X, t)}) \longrightarrow Q(\mathcal{C}(X, B)_{\mathcal{C}(X, s)})$$

with values in $\mathbf{Cat}_{\mathbf{cc}}$, or equivalently a pseudonatural transformation

$$\mathcal{C}(X, A)_{\mathcal{C}(X, t)} \longrightarrow Q(\mathcal{C}(X, B)_{\mathcal{C}(X, s)})$$

with values in \mathbf{Cat} , which in turn amounts to a pseudonatural transformation

$$\mathcal{C}(X, A) \longrightarrow Q(\mathcal{C}(X, B)_{\mathcal{C}(X, s)})$$

equipped with an op-action of $\mathcal{C}(X, t)$. By Yoneda the pseudonatural transformation amounts to an object of $Q(\mathcal{C}(A, B)_{\mathcal{C}(A, s)})$; that is, a morphism $f : A \rightarrow B$ equipped with a 2-cell $\varphi_1 : f \rightarrow sf$ which is idempotent in $\mathcal{C}(A, B)_{\mathcal{C}(A, s)}$, or equivalently which satisfies $\mu f.s\varphi_1.\varphi_1 = \varphi_1$. The op-action consists of a morphism in $Q(\mathcal{C}(A, B)_{\mathcal{C}(A, s)})$ from $(ft, \varphi_1 t)$ to (f, φ_1) , satisfying associativity and unitality conditions. This then

amounts to a 2-cell $\varphi : ft \rightarrow sf$ in \mathcal{K} satisfying in addition to associativity and unitality two further normalization conditions. The unitality condition says that $\varphi.f\eta$ is just φ_1 ; it then turns out that the normalization conditions follow from the single associativity condition $\mu f.s\varphi.\varphi t = \varphi.f\mu$. (Idempotency of φ_1 is then automatic.)

To summarize the situation so far, an object $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ is a monad, such as (A, t) . A morphism from (A, t) to (B, s) is a morphism $f : A \rightarrow B$ in \mathcal{K} equipped with a 2-cell $\varphi : ft \rightarrow sf$ satisfying the associativity condition given above. What finally is a 2-cell between two such morphisms (f, φ) and (g, ψ) ? These should be modifications between the corresponding pseudonatural transformations

$$Q(\mathcal{C}(X, A)_{\mathcal{C}(X, t)}) \longrightarrow Q(\mathcal{C}(X, B)_{\mathcal{C}(X, s)})$$

which reduce to modifications, compatible with the op-actions of t , between pseudonatural transformations

$$\mathcal{C}(X, A) \longrightarrow Q(\mathcal{C}(X, B)_{\mathcal{C}(X, s)})$$

which by Yoneda amount to 2-cells $\rho : f \rightarrow sg$ subject to two conditions stated in the theorem below; one gives compatibility with the op-actions, the other is a normalization condition.

Theorem 5.2. *Let \mathcal{K} be a 2-category in which idempotent 2-cells split. The free completion of \mathcal{K} as \mathbf{Cat}_{cc} -category under bicategorical Kleisli objects, or equivalently the free completion of \mathcal{K} under demi-KL-objects, is the evident 2-category $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ in which*

- (i) *an object is a monad (A, t) in \mathcal{K} ;*
- (ii) *a morphism from (A, t) to (B, s) is a 1-cell $f : A \rightarrow B$ in \mathcal{K} equipped with a 2-cell $\varphi : ft \rightarrow sf$ for which the following diagram commutes;*

$$\begin{array}{ccccc} ftt & \xrightarrow{\varphi t} & sft & \xrightarrow{s\varphi} & ssf \\ f\mu \downarrow & & & & \downarrow \mu f \\ ft & \xrightarrow{\varphi} & sf & & \end{array}$$

- (iii) *a 2-cell from (f, φ) to (g, ψ) is a 2-cell $\rho : f \rightarrow sg$ for which the following diagrams commute.*

$$\begin{array}{ccccc} ft & \xrightarrow{\rho t} & sgt & \xrightarrow{s\psi} & ssg \\ \varphi \downarrow & & & & \downarrow \mu g \\ sf & \xrightarrow{s\rho} & ssg & \xrightarrow{\mu g} & sg \end{array} \quad \begin{array}{ccccc} sg & \xrightarrow{sg\eta} & sgt & \xrightarrow{s\psi} & ssg \\ \rho \uparrow & & & & \downarrow \mu g \\ f & \xrightarrow{\rho} & sg & & \end{array}$$

There is a formal dual of this, involving EM- rather than KL-objects. We write $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ for $\mathbf{KL}_{\text{dm}}(\mathcal{K}^{\text{op}})^{\text{op}}$. This is exactly the 2-category $\mathbf{EM}^w(\mathcal{K})$ of [4].

Corollary 5.3. *If \mathcal{K} is a 2-category in which idempotent 2-cells split, then $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ is the free \mathbf{Cat}_{cc} -completion of \mathcal{K} under demi-EM-objects.*

What about the case of a general 2-category \mathcal{K} ? There is a forgetful 2-functor from the 2-category of \mathbf{Cat}_{cc} -categories with demi-KL-objects to the 2-category of 2-categories, and this forgetful 2-functor has a left biadjoint whose object map can be constructed by first applying Q_* , then the construction given above. We write $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ for the \mathbf{Cat}_{cc} -category obtained by applying this left biadjoint to a 2-category \mathcal{K} , and call it the free \mathbf{Cat}_{cc} -category with demi-KL-objects on \mathcal{K} . An object of $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ is just a demimonad in \mathcal{K} ; we write this as (A, t) , with remaining structure (μ_2, μ_1, μ_0) omitted from the notation. A 1-cell from (A, t) to (B, s) consists of a 1-cell (f, \bar{f}) in $Q_*\mathcal{K}$ equipped with a 2-cell $\varphi : (f, \bar{f})(t, \mu_1) \rightarrow (s, \mu_1)(f, \bar{f})$ in $Q_*\mathcal{K}$ satisfying associativity. (We shall see shortly that a simplification is possible.) A 2-cell from (f, \bar{f}, φ) to (g, \bar{g}, ψ) is a 2-cell $\rho : (f, \bar{f}) \rightarrow (s, \mu_1)(g, \bar{g})$ satisfying the two conditions above. A 2-cell $(f, \bar{f}) \rightarrow (s, \mu_1)(g, \bar{g})$ is a 2-cell $\rho : f \rightarrow sg$ such that $\rho \bar{f} = \rho = \mu_1 g.s\bar{g}.\rho$; the other two conditions are unchanged.

Consider a 1-cell $(f, \bar{f}, \varphi) : (A, t) \rightarrow (B, s)$. Let $\varphi_1 = \varphi.f\eta : f \rightarrow sf$. This clearly defines a 2-cell from $(f, \bar{f}) \rightarrow (s, \mu_1)(f, 1)$, and compatibility with the op-action holds by $\mu f.s\varphi_1.\varphi = \mu f.s\varphi.s\eta.\varphi =$

$\mu f.s\varphi.\varphi t.ft\eta = \varphi.f\mu.ft\eta = \varphi.f\mu_1 = \varphi.f\mu.ft\eta = \mu f.s\varphi.\varphi t.ft\eta = \mu f.s\varphi.\varphi_1 t$ and finally the normalization condition by idempotency of φ_1 ; i.e. $\mu f.s\varphi.sf\eta.\varphi_1 = \mu f.s\varphi.sf\eta.\varphi.f\eta = \mu f.s\varphi.\varphi t.ft\eta.f\eta = \varphi.f\mu.ft\eta.f\eta = \varphi.f\eta = \varphi_1$; thus φ_1 is a 2-cell from (f, \bar{f}, φ) to $(f, 1, \varphi)$.

Similarly, φ_1 is clearly a 2-cell from $(f, 1) \rightarrow (s, \mu_1)(f, \bar{f})$, and compatibility with the op-actions and the normalization condition hold exactly as before, so we have a 2-cell from $(f, 1, \varphi)$ to (f, \bar{f}, φ) , clearly inverse to the previous one.

Thus in our 1-cells, we may as well restrict to those of the form $(f, 1, \varphi)$, which we henceforth write simply as (f, φ) . This gives the following description of $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ for general \mathcal{K} :

Theorem 5.4. *The free \mathbf{Cat}_{cc} -category with demi-KL-objects on a 2-category \mathcal{K} is the evident 2-category $\mathbf{KL}_{\text{dm}}(\mathcal{K})$ in which*

- (i) *an object is a demimonad (A, t) in \mathcal{K} ;*
- (ii) *a morphism from (A, t) to (B, s) is a 1-cell $f : A \rightarrow B$ in \mathcal{K} equipped with a 2-cell $\varphi : ft \rightarrow sf$ for which the following diagrams commute;*

$$\begin{array}{ccc} ft & \xrightarrow{\varphi t} & sft \xrightarrow{s\varphi} ssf \\ f\mu \downarrow & & \downarrow \mu f \\ ft & \xrightarrow{\varphi} & sf \end{array} \quad \begin{array}{ccc} ft & \xrightarrow{\varphi} & sf \\ f\mu_1 \downarrow & \searrow \varphi & \downarrow \mu_1 f \\ ft & \xrightarrow{\varphi} & sf \end{array}$$

- (iii) *a 2-cell from (f, φ) to (g, ψ) is a 2-cell $\rho : f \rightarrow sg$ for which the following diagrams commute.*

$$\begin{array}{ccc} ft & \xrightarrow{\rho t} & sgt \xrightarrow{s\psi} ssg \\ \varphi \downarrow & & \downarrow \mu g \\ sf & \xrightarrow{s\rho} & ssg \xrightarrow{\mu g} sg \end{array} \quad \begin{array}{ccc} sg & \xrightarrow{sg\eta} & sgt \xrightarrow{s\psi} ssg \\ \rho \uparrow & & \downarrow \mu g \\ f & \xrightarrow{\rho} & sg \end{array}$$

The new condition on morphisms says that the following composites

$$ft \xrightarrow{\varphi} sf \xrightarrow{\eta sf} s^2 f \xrightarrow{\mu f} sf \quad \text{and} \quad ft \xrightarrow{ft\eta} ft^2 \xrightarrow{f\mu} ft \xrightarrow{\varphi} sf$$

are both simply equal to φ . This is not automatic, as can be seen by taking $(A, t) = (B, s)$ and $f = 1$: then the identity 2-cell on $1t \rightarrow t1$ does not satisfy this condition unless (A, t) is actually a monad.

Once again, there is a dual result for demi-EM-objects:

Theorem 5.5. *The free \mathbf{Cat}_{cc} -category with demi-EM-objects on a 2-category \mathcal{K} is the evident 2-category $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ in which*

- (i) *an object is a demimonad (A, t) in \mathcal{K} ;*
- (ii) *a morphism from (A, t) to (B, s) is a 1-cell $f : A \rightarrow B$ in \mathcal{K} equipped with a 2-cell $\varphi : sf \rightarrow ft$ for which the following diagrams commute;*

$$\begin{array}{ccc} ssf & \xrightarrow{s\varphi} & sft \xrightarrow{\varphi t} ftt \\ \mu f \downarrow & & \downarrow f\mu \\ sf & \xrightarrow{\varphi} & ft \end{array} \quad \begin{array}{ccc} sf & \xrightarrow{\varphi} & ft \\ \mu_1 f \downarrow & \searrow \varphi & \downarrow f\mu_1 \\ sf & \xrightarrow{\varphi} & ft \end{array}$$

- (iii) *a 2-cell from (f, φ) to (g, ψ) is a 2-cell $\rho : f \rightarrow gt$ for which the following diagrams commute.*

$$\begin{array}{ccc} sf & \xrightarrow{s\rho} & sgt \xrightarrow{\psi t} gtt \\ \varphi \downarrow & & \downarrow g\mu \\ ft & \xrightarrow{\rho t} & gtt \xrightarrow{g\mu} gt \end{array} \quad \begin{array}{ccc} gt & \xrightarrow{\eta gt} & sgt \xrightarrow{\psi t} gtt \\ \rho \uparrow & & \downarrow g\mu \\ f & \xrightarrow{\rho} & gt \end{array}$$

6. FORMAL THEORY OF MONADS

The basic ingredients of the formal theory of monads, as presented in [19], are as follows. For any 2-category \mathcal{K} , there is a 2-category $\mathbf{Mnd}(\mathcal{K})$ whose objects are monads in \mathcal{K} , and a fully faithful 2-functor $\text{Id} : \mathcal{K} \rightarrow \mathbf{Mnd}(\mathcal{K})$, sending an object of \mathcal{K} to the identity monad on that object. This 2-functor has a right adjoint if and only if \mathcal{K} has Eilenberg-Moore objects; the right adjoint then takes a monad to its Eilenberg-Moore object. Furthermore, there is a monad \mathbf{Mnd} on the category $\mathbf{2-Cat}$ of 2-categories and 2-functors, and the endofunctor part of \mathbf{Mnd} sends an object \mathcal{K} to $\mathbf{Mnd}(\mathcal{K})$, while $\text{Id} : \mathcal{K} \rightarrow \mathbf{Mnd}(\mathcal{K})$ is the component at \mathcal{K} of the unit of the monad. An object of $\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{K}))$ — that is, a monad in $\mathbf{Mnd}(\mathcal{K})$ — is the same thing as a distributive law, and the multiplication $\text{Comp} : \mathbf{Mnd}(\mathbf{Mnd}(\mathcal{K})) \rightarrow \mathbf{Mnd}(\mathcal{K})$ of the monad \mathbf{Mnd} sends a distributive law to the induced composite monad.

In the sequel [14] to [19], a variant $\mathbf{EM}(\mathcal{K})$ of $\mathbf{Mnd}(\mathcal{K})$ was proposed, with the same objects and 1-cells as $\mathbf{Mnd}(\mathcal{K})$, but with a more general notion of 2-cell. Once again, this is the object-part of a monad on $\mathbf{2-Cat}$, and the unit $\text{Id} : \mathcal{K} \rightarrow \mathbf{EM}(\mathcal{K})$ has a right adjoint if and only if \mathcal{K} has Eilenberg-Moore objects; but this time there is a conceptual explanation: $\mathbf{EM}(\mathcal{K})$ is the free completion of \mathcal{K} under Eilenberg-Moore objects. From this universal property of $\mathbf{EM}(\mathcal{K})$, it follows immediately that $\text{Id} : \mathcal{K} \rightarrow \mathbf{EM}(\mathcal{K})$ will have a right adjoint if and only if \mathcal{K} has Eilenberg-Moore objects; in particular, since $\mathbf{EM}(\mathcal{K})$ has Eilenberg-Moore objects, we obtain the multiplication $\text{Comp} : \mathbf{EM}(\mathbf{EM}(\mathcal{K})) \rightarrow \mathbf{EM}(\mathcal{K})$.

6.1. Wreaths. An object of $\mathbf{EM}(\mathbf{EM}(\mathcal{K}))$ — that is, a monad in $\mathbf{EM}(\mathcal{K})$ — is more general than a monad in $\mathbf{Mnd}(\mathcal{K})$, because of the more general 2-cells in $\mathbf{EM}(\mathcal{K})$. Thus we obtain a more general notion of distributive law, called a *wreath* in [14].

When it comes to the weak version, we have in place of $\mathbf{EM}(\mathcal{K})$ our weak version $\mathbf{EM}_{\text{dm}}(\mathcal{K})$, the free completion of \mathcal{K} under demi-EM-objects. Once again, we can draw various immediate conclusions from this universal property of $\mathbf{EM}_{\text{dm}}(\mathcal{K})$; many of these were given concrete proofs in [4], using the concrete description of $\mathbf{EM}_{\text{dm}}(\mathcal{K})$. For instance, writing once again $\text{Id}_{\mathcal{K}} : \mathcal{K} \rightarrow \mathbf{EM}_{\text{dm}}(\mathcal{K})$ for the inclusion we have:

Theorem 6.1. *For any 2-category \mathcal{K} , the inclusion $\text{Id}_{Q_*\mathcal{K}} : Q_*\mathcal{K} \rightarrow \mathbf{EM}_{\text{dm}}(Q_*\mathcal{K}) \cong \mathbf{EM}_{\text{dm}}(\mathcal{K})$ has a right biadjoint if and only if \mathcal{K} has demi-EM objects. In particular, the inclusion $\text{Id}_{\mathcal{K}} : \mathcal{K} \rightarrow \mathbf{EM}_{\text{dm}}(\mathcal{K})$ has a right biadjoint whenever \mathcal{K} has bicategorical EM-objects and idempotent 2-cells split.*

PROOF: Note that for any object X and any demimonad (A, t) in \mathcal{K} , both categories $Q_*\mathcal{K}(X, A)^{Q_*\mathcal{K}(X, t)} \cong \mathbf{Mnd}(Q_*\mathcal{K})((X, 1), (A, t))$ and $\mathbf{EM}_{\text{dm}}(\mathcal{K})((X, 1), (A, t))$ are isomorphic. Hence the claim follows from the definition of the demi-EM object via the pseudonatural equivalence $Q_*\mathcal{K}(X, A^{(t)}) \simeq Q_*\mathcal{K}(X, A)^{Q_*\mathcal{K}(X, t)}$. \square

In particular, the locally Cauchy complete 2-category $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ does have demi-EM-objects, and so the inclusion $\mathbf{EM}_{\text{dm}}(\mathcal{K}) \rightarrow \mathbf{EM}_{\text{dm}}(\mathbf{EM}_{\text{dm}}(\mathcal{K}))$ does have a right biadjoint, which sends demimonads $((A, t), (s, \lambda))$ in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ to demimonads (A, st) in \mathcal{K} . We might call a demimonad in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ a *demiwreath* in \mathcal{K} . As an instance of the preceding theorem, every demiwreath induces a composite demimonad. This is a (minor) generalization of one direction of [4, Theorem 2.3], in that it deals from the outset with demimonads rather than monads.

The demi-EM object of the composite demimonad (A, st) is defined via the pseudonatural equivalence

$$Q_*\mathcal{K}(X, A^{(st)}) \simeq \mathbf{EM}_{\text{dm}}(\mathcal{K})((X, 1), (A, st)) \simeq \mathbf{EM}_{\text{dm}}(\mathbf{EM}_{\text{dm}}(\mathcal{K}))(((X, 1), 1), ((A, t), (s, \lambda))).$$

On the other hand, as said above, whenever demi-EM objects exist in \mathcal{K} , $\text{Id}_{Q_*\mathcal{K}} : Q_*\mathcal{K} \rightarrow \mathbf{EM}_{\text{dm}}(\mathcal{K})$ has a right biadjoint J sending an object (A, t) of $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ (i.e. demimonad in \mathcal{K}) to the demi-EM object $A^{(t)}$. It induces a pseudofunctor $\mathbf{Mnd}(J) : \mathbf{Mnd}(\mathbf{EM}_{\text{dm}}(\mathcal{K})) \rightarrow \mathbf{Mnd}(Q_*\mathcal{K})$, taking a demimonad $((A, t), (s, \lambda))$ in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$ to the demimonad $(J(A, t), J(s, \lambda)) = (A^{(t)}, J(s, \lambda))$ in \mathcal{K} . The demi-EM object of this latter monad is defined via the pseudonatural equivalence

$$\begin{aligned} Q_*\mathcal{K}(X, (A^{(t)})^{(J(s, \lambda))}) &\simeq \mathbf{Mnd}(Q_*\mathcal{K})((X, 1), \mathbf{Mnd}(J)((A, t), (s, \lambda))) \\ &\simeq \mathbf{Mnd}(\mathbf{EM}_{\text{dm}}(\mathcal{K}))(((X, 1), 1), ((A, t), (s, \lambda))) \\ &\cong \mathbf{EM}_{\text{dm}}(\mathbf{EM}_{\text{dm}}(\mathcal{K}))(((X, 1), 1), ((A, t), (s, \lambda))). \end{aligned}$$

Thus we conclude that, whenever demi-EM objects exist in \mathcal{K} , $A^{(st)}$ and $(A^{(t)})^{(J(s,\lambda))}$ are equivalent objects of $Q_*\mathcal{K}$. This extends some observations in [4, Proposition 3.7].

6.2. Lifting. Another key aspect of the formal theory of monads is that, for a 2-category \mathcal{K} with Eilenberg-Moore objects, monad morphisms from $(f, \varphi) : (A, t) \rightarrow (B, s)$ are in bijection with morphisms $f : A \rightarrow B$ equipped with liftings

$$\begin{array}{ccc} A^t & \xrightarrow{\bar{f}} & B^s \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

of f to the Eilenberg-Moore objects. (More generally this is true provided that the Eilenberg-Moore objects A^t and B^s exist.) Similarly, for two such morphisms $(f, \varphi), (g, \psi) : (A, t) \rightarrow (B, s)$, a 2-cell $\rho : f \rightarrow g$ gives a 2-cell $(f, \varphi) \rightarrow (g, \psi)$ in $\mathbf{Mnd}(\mathcal{K})$ if and only if it lifts to a 2-cell $\bar{f} \rightarrow \bar{g}$ between the corresponding lifted morphisms from A^t to B^s . On the other hand a 2-cell in $\mathbf{EM}(\mathcal{K})$ from (f, φ) to (g, ψ) is just an arbitrary 2-cell $\bar{f} \rightarrow \bar{g}$. There are analogues of this for $\mathbf{EM}_{\text{dm}}(\mathcal{K})$.

In the previous section we have seen that, whenever demi-EM objects exist in \mathcal{K} (equivalently, bicategorical EM objects exist in $Q_*\mathcal{K}$), $\text{Id}_{Q_*\mathcal{K}} : Q_*\mathcal{K} \rightarrow \mathbf{EM}_{\text{dm}}(\mathcal{K})$ possesses a right biadjoint J with object map $(B, t) \mapsto B^{(t)}$. The counit of the biadjunction is given by the 1-cell $(u, \psi) : (B^{(t)}, 1) \rightarrow (B, t)$ from Section 4.2, for any demimonad (B, t) , and the iso 2-cell

$$(4) \quad \begin{array}{ccc} (A^{(s)}, 1) & \xrightarrow{(J(g,\lambda), 1)} & (B^{(t)}, 1) \\ (u, \psi) \downarrow & \Downarrow \xi & \downarrow (u, \psi) \\ (A, s) & \xrightarrow{(g, \lambda)} & (B, t) \end{array}$$

for any demimonad morphism (i.e. 1-cell in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$) $(g, \lambda) : (A, s) \rightarrow (B, t)$. Explicitly, such an iso 2-cell is given by 2-cells $\xi : uJ(g, \lambda) \rightarrow gu$ and $\xi' : gu \rightarrow uJ(g, \lambda)$ in \mathcal{K} such that the normalization conditions

$$\begin{array}{ccccc} uJ(g, \lambda) & \xrightarrow{\xi} & gu & & gu & \xrightarrow{\xi'} & uJ(g, \lambda) & & gu & \xrightarrow{\xi'} & uJ(g, \lambda) & & uJ(g, \lambda) & \xrightarrow{\xi} & gu \\ & \searrow \xi & \downarrow \lambda_1 & & \searrow \xi' & \downarrow \psi_1 J(g, \lambda) & & \searrow \xi & \downarrow \lambda_1 & & \searrow \xi' & \downarrow \psi_1 J(g, \lambda) & & \searrow \xi & \downarrow \lambda_1 \\ & & gu & & & uJ(g, \lambda) & & & gu & & & uJ(g, \lambda) & & & gu \end{array}$$

and the ‘ t -linearity’ condition

$$\begin{array}{ccc} tuJ(g, \lambda) & \xrightarrow{t\xi} & tgu \\ \psi J(g, \lambda) \downarrow & & \downarrow \lambda u \\ uJ(g, \lambda) & \xrightarrow{\xi} & gu \end{array}$$

hold, where we introduced the idempotent 2-cells $\psi_1 := \psi \cdot \eta u$ and $\lambda_1 := g\psi \cdot \lambda u \cdot \eta gu$, corresponding to the t -demialgebras (u, ψ) and $(gu, g\psi \cdot \lambda u)$, respectively. Note that in particular $g\psi_1 \cdot \xi = \xi = \xi \cdot \psi_1 J(g, \lambda)$ and $\xi' \cdot g\psi_1 = \xi' = \psi_1 J(g, \lambda) \cdot \xi'$.

In what follows we show that the map $(g, \lambda) \mapsto J(g, \lambda)$ provides the object map of an equivalence between the hom category of some 2-category of monads; and an appropriately defined category of liftings for demi-EM objects $A^{(s)} \rightarrow B^{(t)}$.

Lemma 6.2. *Consider demimonads (A, s) and (B, t) in a 2-category \mathcal{K} in which demi-EM objects exist. If for some 1-cells $g : A \rightarrow B$ and $h : A^{(s)} \rightarrow B^{(t)}$ there exist 2-cells $\zeta : uh \rightarrow gu$ and $\zeta' : gu \rightarrow uh$ such that*

$\zeta' \cdot \zeta = \psi_1 h$ and the normalization conditions $g\psi_1 \cdot \zeta = \zeta = \zeta \cdot \psi_1 h$ and $\zeta' \cdot g\psi_1 = \zeta' = \psi_1 h \cdot \zeta'$ hold, then there exists a demimonad morphism $(g, \lambda) : (A, s) \rightarrow (B, t)$ such that $J(g, \lambda)$ is isomorphic to h .

PROOF: The requested 1-cell $(g, \lambda) : (A, s) \rightarrow (B, t)$ is constructed by introducing λ as the composite

$$tg \xrightarrow{tg\eta} tgs \xrightarrow{tg\chi'} tgu f \xrightarrow{t\zeta' f} tuh f \xrightarrow{\psi h f} uh f \xrightarrow{\zeta f} gu f \xrightarrow{gx} gs,$$

where the notations from Section 4.2 are used. The corresponding idempotent 2-cell $\lambda_1 : gu \rightarrow gu$ comes out as $\zeta \cdot \zeta'$. For the induced t -demi-action $g\psi \cdot \lambda u = \zeta \cdot \psi h \cdot t\zeta' : tgu \rightarrow gu$, both ζ and ζ' are morphisms of t -demi-algebras. Hence together with the canonical 2-cells $\xi : uJ(g, \lambda) \rightarrow gu$ and $\xi' : gu \rightarrow uJ(g, \lambda)$, they induce t -demi-algebra morphisms $\xi' \cdot \zeta : uh \rightarrow uJ(g, \lambda)$ and $\zeta' \cdot \xi : uJ(g, \lambda) \rightarrow uh$. These are subject to the normalization conditions in Section 4.2 hence by universality of (u, ψ) give rise to mutually inverse isomorphisms $\alpha : h \rightarrow J(g, \lambda)$ and $\alpha' : J(g, \lambda) \rightarrow h$. \square

By naturality of ξ , for any 2-cell $\varrho : (g, \lambda) \rightarrow (g', \lambda')$ in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$, the 2-cell $J(\varrho) : J(g, \lambda) \rightarrow J(g', \lambda')$ renders commutative

$$(5) \quad \begin{array}{ccc} uJ(g, \lambda) & \xrightarrow{\xi} & gu \\ \downarrow uJ(\varrho) & & \downarrow \varrho u \\ uJ(g', \lambda') & \xrightarrow{\xi} & g'u \end{array} \quad \text{equivalently,} \quad \begin{array}{ccc} gu & \xrightarrow{\xi'} & uJ(g, \lambda) \\ \downarrow \varrho u & & \downarrow uJ(\varrho) \\ g'su & & g'u \\ \downarrow g'\psi & & \downarrow \xi' \\ g'u & \xrightarrow{\xi'} & uJ(g', \lambda') \end{array}$$

The above two equivalent forms of the same equality provide us with two symmetrical choices how to define a lifting of a 2-cell in \mathcal{K} for demi-EM objects: we can require either one to take a particularly simple form.

Lemma 6.3. *Let \mathcal{K} be a 2-category in which demi-EM objects exist and (g, λ) and (g', λ') be demimonad morphisms $(A, s) \rightarrow (B, t)$ in \mathcal{K} .*

(1) *For a 2-cell $\omega : g \rightarrow g'$, the following are equivalent.*

(i) *the following diagram commutes;*

$$\begin{array}{ccccc} tg & \xrightarrow{t\omega} & tg' & & \\ \downarrow \lambda & & \downarrow \lambda' & & \\ gs & \xrightarrow{\omega s} g's & \xrightarrow{\eta g's} tg's & \xrightarrow{\lambda' s} g's^2 & \xrightarrow{g'\mu} g's \end{array}$$

(ii) $\lambda'_1 \cdot \omega u$ is a t -demi-algebra morphism $(gu, g\psi \cdot \lambda u) \rightarrow (g'u, g'\psi \cdot \lambda' u)$;

(iii) $\lambda' \cdot \eta g' \cdot \omega : g \rightarrow g's$ is a 2-cell $(g, \lambda) \rightarrow (g', \lambda')$ in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$;

(iv) *there is a 2-cell $\bar{\omega} : J(g, \lambda) \rightarrow J(g', \lambda')$ such that the following diagram commutes.*

$$\begin{array}{ccc} gu & \xrightarrow{\omega u} & g'u \\ \downarrow \xi' & & \downarrow \xi' \\ uJ(g, \lambda) & \xrightarrow{u\bar{\omega}} & uJ(g', \lambda') \end{array}$$

If these assertions hold then $\bar{\omega} = J(\lambda' \cdot \eta g' \cdot \omega)$.

(2) *For a 2-cell $\omega : g \rightarrow g'$, the following are equivalent.*

(i) the following diagram commutes;

$$\begin{array}{ccccccc}
 tg & \xrightarrow{t\eta g} & t^2g & \xrightarrow{t\lambda} & tgs & \xrightarrow{t\omega s} & tg's \\
 \downarrow \lambda & & & & & & \downarrow \lambda's \\
 & & & & & & g's^2 \\
 & & & & & & \downarrow g'\mu \\
 gs & \xrightarrow{\omega s} & & & g's & &
 \end{array}$$

- (ii) $\omega u.\lambda_1$ is a t -demialgebra morphism $(gu, g\psi.\lambda u) \rightarrow (g'u, g'\psi.\lambda'u)$;
- (iii) $\omega s.\lambda.\eta g : g \rightarrow g's$ is a 2-cell $(g, \lambda) \rightarrow (g', \lambda')$ in $\mathbf{EM}_{\text{dm}}(\mathcal{K})$;
- (iv) there is a 2-cell $\vec{\omega} : J(g, \lambda) \rightarrow J(g', \lambda')$ such that the following diagram commutes.

$$\begin{array}{ccc}
 uJ(g, \lambda) & \xrightarrow{u\vec{\omega}} & uJ(g', \lambda') \\
 \downarrow \xi & & \downarrow \xi \\
 gu & \xrightarrow{\omega u} & g'u
 \end{array}$$

If these assertions hold then $\vec{\omega} = J(\omega s.\lambda.\eta g)$.

PROOF: Consider first part (1). The diagram in part (i) is equivalent to t -linearity of the 2-cell in part (ii). If this holds then the normalization conditions on the 2-cell in part (ii) are automatic. Thus (i) \Leftrightarrow (ii). Similarly, the diagram in part (i) is equivalent to the first diagram in Theorem 5.5 (iii) for taking the 2-cell in part (iii) as “ g ”. If this holds then the second condition in Theorem 5.5 (iii) is automatic. Thus (i) \Leftrightarrow (iii). If the assertion in part (iii) holds, then we can obtain $\vec{\omega}$ in part (iv) by applying the pseudofunctor J to the 2-cell in part (iii). The diagram in part (iv) is then just the second diagram in (5) for the 2-cell in part (iii). Finally, assume that assertion (iv) holds. Then $\lambda'_1.\omega u = \xi.\xi'.\omega u = \xi.u\vec{\omega}.\xi'$ is evidently a morphism of t -demialgebras hence also (ii) holds.

Part (2) is proven similarly, using the first diagram in (5) instead of the second one. \square

Corollary 6.4. *Let \mathcal{K} be a 2-category in which demi-EM objects exist and (A, s) and (B, t) be demimonads in \mathcal{K} .*

(1) *The following categories are equivalent.*

- (i) *The category whose objects are quadruples $(g : A \rightarrow B, h : A^{(s)} \rightarrow B^{(t)}, \zeta : uh \rightarrow gu, \zeta' : gu \rightarrow uh)$ such that $\zeta' \cdot \zeta = \psi_1 h$ and the normalization conditions $g\psi_1 \cdot \zeta = \zeta = \zeta \cdot \psi_1 h$ and $\zeta' \cdot g\psi_1 = \zeta' = \psi_1 h \cdot \zeta'$ hold. Morphisms $(g, h, \zeta, \zeta') \rightarrow (\tilde{g}, \tilde{h}, \tilde{\zeta}, \tilde{\zeta}')$ are pairs $(\omega : g \rightarrow \tilde{g}, \varphi : h \rightarrow \tilde{h})$ such that $u\varphi \cdot \zeta' = \tilde{\zeta}' \cdot \omega u$.*
- (ii) *The category whose objects are demimonad morphisms $(A, s) \rightarrow (B, t)$ and morphisms $(g, \lambda) \rightarrow (g', \lambda')$ are 2-cells $\omega : g \rightarrow g'$ rendering commutative the diagram in Lemma 6.3 (1)(i).*

(2) *The following categories are equivalent.*

- (i) *The category whose objects are quadruples $(g : A \rightarrow B, h : A^{(s)} \rightarrow B^{(t)}, \zeta : uh \rightarrow gu, \zeta' : gu \rightarrow uh)$ such that $\zeta' \cdot \zeta = \psi_1 h$ and the normalization conditions $g\psi_1 \cdot \zeta = \zeta = \zeta \cdot \psi_1 h$ and $\zeta' \cdot g\psi_1 = \zeta' = \psi_1 h \cdot \zeta'$ hold. Morphisms $(g, h, \zeta, \zeta') \rightarrow (\tilde{g}, \tilde{h}, \tilde{\zeta}, \tilde{\zeta}')$ are pairs $(\omega : g \rightarrow \tilde{g}, \varphi : h \rightarrow \tilde{h})$ such that $\zeta \cdot u\varphi = \omega u \cdot \zeta$.*
- (ii) *The category whose objects are demimonad morphisms $(A, s) \rightarrow (B, t)$ and morphisms $(g, \lambda) \rightarrow (g', \lambda')$ are 2-cells $\omega : g \rightarrow g'$ rendering commutative the diagram in Lemma 6.3 (2)(i).*

PROOF: By Lemma 6.3, there are fully faithful functors from the categories in parts (ii) to the respective categories in part (i). They are essentially surjective on the objects by Lemma 6.2: for any object $(g : A \rightarrow B, h : A^{(s)} \rightarrow B^{(t)}, \zeta : uh \rightarrow gu, \zeta' : gu \rightarrow uh)$ in part (i), the isomorphism $h \rightarrow J(g, \lambda)$ in Lemma 6.2 and the identity morphism $g \rightarrow g$ constitute an isomorphism in the category in question. \square

The categories in the above corollary are hom categories in evident 2-categories. Both parts (ii) amount to extensions of $\mathbf{Mnd}(\mathcal{K})$ in two inequivalent ways.

As we recalled earlier, a distributive law is in fact a monad in $\mathbf{Mnd}(\mathcal{K})$. Weak distributive laws in [21] are not the same as monads in either generalization of $\mathbf{Mnd}(\mathcal{K})$ in the above corollary. However, following the lines in [6], they can be described as compatible pairs of monads in both of them.

APPENDIX A. FREE $\mathbf{Cat}_{\mathbf{cc}}$ -COMPLETIONS

The classical theory of weighted colimits and colimit completions can be found in [12]. It applies for categories enriched in a complete and cocomplete symmetric monoidal closed category \mathcal{V} . Here we adapt it to deal with the case of categories enriched over the Cartesian closed category $\mathbf{Cat}_{\mathbf{cc}}$, which is neither complete nor cocomplete. It is, however, complete and cocomplete as a bicategory, and this will be the basis of our approach.

We write $J : \mathbf{Cat}_{\mathbf{cc}} \rightarrow \mathbf{Cat}$ for the fully faithful inclusion; it has a left biadjoint Q . For 2-categories \mathcal{A} and \mathcal{B} we write $[\mathcal{A}, \mathcal{B}]$ for the usual 2-category of 2-functors from \mathcal{A} to \mathcal{B} , with 2-natural transformations as 1-cells and modifications as 2-cells. We write $\mathbf{Hom}(\mathcal{A}, \mathcal{B})$ for the 2-category of pseudofunctors, pseudonatural transformations, and modifications, and $\mathbf{Ps}(\mathcal{A}, \mathcal{B})$ for the 2-category of 2-functors, pseudonatural transformations, and modifications.

Remark A.1. Let $F : \mathcal{A} \rightarrow \mathbf{Cat}_{\mathbf{cc}}$ be a pseudofunctor. Then $JF : \mathcal{A} \rightarrow \mathbf{Cat}$ is also a pseudofunctor. It is pseudonaturally equivalent to a 2-functor $G : \mathcal{A} \rightarrow \mathbf{Cat}$; but any category equivalent to a Cauchy complete one is itself Cauchy complete, and so G lands in $\mathbf{Cat}_{\mathbf{cc}}$, and can be written as JH for some 2-functor $H : \mathcal{A} \rightarrow \mathbf{Cat}_{\mathbf{cc}}$, which is then pseudonaturally equivalent to F . Thus the 2-categories $\mathbf{Ps}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ and $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ are biequivalent.

Recall that for pseudofunctors $S : \mathcal{D} \rightarrow \mathcal{K}$ and $F : \mathcal{D}^{\mathrm{op}} \rightarrow \mathbf{Cat}$, the bicolimit $F * S$ is defined by a pseudonatural equivalence

$$\mathcal{K}(F * S, A) \simeq \mathbf{Hom}(\mathcal{D}^{\mathrm{op}}, \mathbf{Cat})(F, \mathcal{K}(S, A)).$$

(In fact it does no harm to suppose that the map from left to right is strictly natural in A , and so is induced by a pseudonatural $F \rightarrow \mathcal{K}(S, F * S)$, but the inverse equivalence, going from right to left, will still only be pseudonatural.)

If \mathcal{K} is a $\mathbf{Cat}_{\mathbf{cc}}$ -category, then we may choose to restrict to the case where \mathcal{D} is a $\mathbf{Cat}_{\mathbf{cc}}$ -category and F lands in $\mathbf{Cat}_{\mathbf{cc}}$.

Proposition A.2. *For a small $\mathbf{Cat}_{\mathbf{cc}}$ -category \mathcal{A} , the 2-category $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ is in fact a $\mathbf{Cat}_{\mathbf{cc}}$ -category with all bicolimits.*

PROOF: The existence of bicolimits follows from the fact that the fully faithful inclusion of $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ in $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat})$ has a left biadjoint.

The idempotent splittings can be computed pointwise. □

Let Φ be a class of $\mathbf{Cat}_{\mathbf{cc}}$ -weights, and \mathcal{A} a small $\mathbf{Cat}_{\mathbf{cc}}$ -category. Write $\Phi(\mathcal{A})$ for the closure in $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ of the representables under Φ -bicolimits. (This can be formed as the intersection of all full subcategories containing the representables and closed under Φ -bicolimits; since $\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ is such a subcategory, and the intersection of any collection of such subcategories is one, the intersection clearly has the desired properties. It can also be formed via a transfinite induction.)

We shall write $W : \Phi(\mathcal{A}) \rightarrow \mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}_{\mathbf{cc}})$ for the inclusion, and $Y : \mathcal{A} \rightarrow \Phi(\mathcal{A})$ for the restricted Yoneda embedding. We wish to prove that $\Phi(\mathcal{A})$ is the free completion of \mathcal{A} under Φ -bicolimits, in the sense that for any $\mathbf{Cat}_{\mathbf{cc}}$ -category \mathcal{K} with Φ -bicolimits, composition with Y induces a biequivalence

$$\Phi\text{-Coc}(\Phi(\mathcal{A}), \mathcal{K}) \simeq \mathbf{Hom}(\mathcal{A}, \mathcal{K})$$

of $\mathbf{Cat}_{\mathbf{cc}}$ -categories.

Our first result holds by definition of $\Phi(\mathcal{A})$:

Proposition A.3. *$\Phi(\mathcal{A})$ has Φ -bicolimits, preserved by W .* □

From now on we shall fix a $\mathbf{Cat}_{\mathbf{cc}}$ -category \mathcal{K} with Φ -colimits.

Proposition A.4. *For any pseudofunctor $F : \mathcal{A} \rightarrow \mathcal{K}$, the (pointwise) left Kan extension $\text{Lan}_Y F : \Phi(\mathcal{A}) \rightarrow \mathcal{K}$ exists.*

PROOF: The formula for the pointwise left Kan extension is

$$(\text{Lan}_Y F)X = X * F$$

so we are to show that the bicolimit on the right exists for all $X \in \Phi(\mathcal{A})$.

Let \mathcal{B} be the full sub- \mathbf{Cat}_{cc} -category of $\Phi(\mathcal{A})$ consisting of those objects X for which $X * F$ does exist. Certainly \mathcal{B} contains the representables, since $\mathcal{A}(-, A) * F$ can be taken to be FA . On the other hand, if $S : \mathcal{D} \rightarrow \mathcal{B}$ and $\varphi : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Cat}_{\text{cc}}$ is in Φ , then each $SD * F$ exists, so we may write $S * F$ for the pseudofunctor sending D to $SD * F$, and now $\varphi * (S * F)$ exists, since \mathcal{K} has Φ -weighted bicolimits. But $\varphi * (S * F) \simeq (\varphi * S) * F$, and so $\varphi * S$ is also in \mathcal{B} . Thus \mathcal{B} contains the representables and is closed under Φ -weighted bicolimits, so must be all of $\Phi(\mathcal{A})$. \square

Proposition A.5. *In the setting of the previous proposition, $\text{Lan}_Y F$ is Φ -cocontinuous.*

PROOF: Let $S : \mathcal{D} \rightarrow \Phi(\mathcal{A})$ and $\varphi : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Cat}_{\text{cc}}$, with $\varphi \in \Phi$. We must show that $(\text{Lan}_Y F)(\varphi * S) \simeq \varphi * (\text{Lan}_Y F)S$. But

$$(\text{Lan}_Y F)(\varphi * S) = (\varphi * S) * F \simeq \varphi * (S * F) = \varphi * (\text{Lan}_Y F)S$$

hence the result. \square

It now follows that the 2-functor

$$\Phi\text{-Coc}(\Phi(\mathcal{A}), \mathcal{K}) \rightarrow \text{Hom}(\mathcal{A}, \mathcal{K})$$

given by restriction along Y has a left biadjoint given by left Kan extension along $Y : \mathcal{A} \rightarrow \Phi(\mathcal{A})$. The component at $F : \mathcal{A} \rightarrow \mathcal{K}$ of the unit is the canonical map $F \rightarrow \text{Lan}_Y(F)Y$, which is an equivalence since Y is one. (More concretely, $\text{Lan}_Y(F)YA = \text{Lan}_Y(F)\mathcal{A}(-, A) \simeq \mathcal{A}(-, A) * F \simeq FA$, and so $\text{Lan}_Y(F)Y \simeq F$.) Thus it remains only to show that the counit is also an equivalence, which amounts to

Proposition A.6. *If $G : \Phi(\mathcal{A}) \rightarrow \mathcal{K}$ is Φ -cocontinuous, then the canonical map $\text{Lan}_Y(GY) \rightarrow G$ is an equivalence.*

PROOF: Let \mathcal{B} be the full subcategory of $\Phi(\mathcal{A})$ consisting of those objects X , for which $\text{Lan}_Y(GY)X \rightarrow GX$ is an equivalence; in other words, for which $X * GY \rightarrow GX$ is an equivalence. Then $\mathcal{A}(-, A) * GY \simeq GYA = G\mathcal{A}(-, A)$, and so \mathcal{B} contains the representables. Suppose that $S : \mathcal{D} \rightarrow \Phi(\mathcal{A})$ lands in \mathcal{B} , and that $\varphi : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Cat}_{\text{cc}}$ is in Φ . Then

$$\begin{aligned} (G \text{ is } \Phi\text{-cocontinuous}) & \quad G(\varphi * S) \simeq \varphi * GS \\ (S \text{ lands in } \mathcal{B}) & \quad \simeq \varphi * (S * GJ) \\ (\text{associativity of } *) & \quad \simeq (\varphi * S) * GJ \end{aligned}$$

and so $\varphi * S$ is also in \mathcal{B} , and thus \mathcal{B} is closed under Φ -colimits. Thus \mathcal{B} is all of $\Phi(\mathcal{A})$. \square

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